Low linear orderings

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Abstract
We say that $L$ is weakly $\eta$-like if $L/\sim$ is isomorphic to the natural ordering of rational numbers. We construct a computable presentation of any low weakly $\eta$-like linear ordering with no strongly $\eta$-like interval. Also we prove that there exists a noncomputably presentable low2 weakly $\eta$-like linear ordering with no strongly $\eta$-like interval.

Keywords: Computability, linear ordering, computable presentations.

1 Introduction
It is known \cite{4} that any low presentable Boolean algebra (i.e. with $X$-computable presentation such that $X' \leq_T \emptyset'$) have a computable copy. This is not true for linear orderings. Namely, C. G. Jockusch and R. I. Soare \cite{9} proved that there exists a low linear ordering with no computable copy. In other hand, R. G. Downey and M. F. Moses \cite{5} stated that any low discrete linear ordering has a computable copy (a linear ordering is discrete if every element has an immediate successor and every element has an immediate predecessor). The following question initiated the study of low linear orderings.

QUESTION 1 (R. G. Downey \cite{3})
Is there a property of order types which guarantees that if a linear ordering $L$ is low and $P(L)$ then $L$ has a computable copy?

DEFINITION 1
A linear ordering is $X$-computable if its domain is an $X$-computable set and its ordering relation is an $X$-computable relation. A $\emptyset$-computable linear ordering is computable.

DEFINITION 2
An $X$-computable linear ordering is low$_n$ if $X^{(n)} \leq_T \emptyset^{(n)}$, where $A^{(n)}$ is the $n$-th jump of $A$. A low$_1$ linear ordering is low.

For any linear ordering $L$ and any $x, y \in |L|$, we define

- the interval $[x, y]_L = \{z | x \leq_L z \leq_L y\}$,
- the successor relation $S_L(x, y) = (x <_L y) \& ([x, y]_L = \{x, y\})$,
- the block relation $F_L(x, y) = (x <_L y) \& ([x, y]_L < \infty)$,
- the equivalent relation $x \sim y = F_L(x, y) \& F_L(y, x)$
- the equivalent class $[x]_L = \{y | y \sim x\}$.

DEFINITION 3
A linear ordering is called quasidiscrete if every equivalent class is either one-element or infinite, i.e. for any $x$, $|[x]| = 1$ or $|[x]| = \infty$. 

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It is obvious that every discrete linear ordering is quasidiscrete and there exists a quasidiscrete linear ordering which is not discrete. Therefore, the following result generalizes the above-mentioned result of R. G. Downey and M. F. Moses.

**Theorem 1** (P. E. Alaev, A. N. Frolov and J. Thurber; A. N. Frolov)

Every low₂ quasidiscrete linear ordering is $\emptyset''$-isomorphic to a computable one (if the ordering is discrete then the isomorphism is $\emptyset''$-computable). On the other hand, there exists a low₃ discrete linear ordering without computable copy.

**Definition 4**

An infinite linear ordering $L$ is strongly $\eta$-like if there is a natural number $k$ such that $|[x]_L| \leq k$ for any $x \in |L|$. It is obvious that there is no discrete linear ordering which is strongly $\eta$-like. Also obviously, a quasidiscrete strongly $\eta$-like linear ordering is dense, i.e. its order type is either $\eta$, or $1+\eta$, or $\eta+1$, or $1+\eta+1$.

**Theorem 2** (A. N. Frolov)

Any low strongly $\eta$-like linear ordering is $\emptyset''$-isomorphic to a computable one.

Recently, we proved in [6] that any low strongly $\eta$-like linear ordering has a computable copy via a $\emptyset'$-computable isomorphism (see below Theorem 3).

**Definition 5**

A linear ordering $L$ is called $k$-quasidiscrete if for any $x \in |L|$ either $|[x]_L| \leq k$ or $|[x]_L| = \infty$.

**Definition 6**

A linear ordering $L$ is called $k$-discrete if for any $x \in |L|$ either $|[x]_L| \leq k$ or $[x]_L \cong \omega^* + \omega$.

Obviously, the class of all 0-discrete linear orderings is exactly the class of all discrete linear orderings; the class of all 1-quasidiscrete linear orderings is exactly the class of all quasidiscrete linear orderings. Also it is easy to see that any strongly $\eta$-like linear ordering is $k$-discrete for some $k$. It is easy to see that for any $k$ there exists a $k$-quasidiscrete linear ordering which is not strongly $\eta$-like and not $k$-discrete. Therefore, at this moment the following result is the best known result on lowly linear orderings.

**Theorem 3** ([6])

Every low $k$-quasidiscrete linear ordering is $\emptyset''$-isomorphic to a computable one. Moreover,

1. if the ordering is $k$-discrete, then the isomorphism is $\emptyset''$-computable,
2. if the ordering is strongly $\eta$-like, then the isomorphism is $\emptyset'$-computable.

## 2 Weakly $\eta$-like linear orderings

**Definition 7**

A linear ordering $L$ is called weakly $\eta$-like if $L/\sim \cong \eta$.

Weakly $\eta$-like linear ordering is $\eta$-like if $[x]_L$ is finite for all $x$. The class of $\eta$-like linear orderings is an important subclass of the class of weakly $\eta$-like. In the study of lowly strongly $\eta$-like linear orderings the following definition is useful.
DEFINITION 8
A function \( f : A \rightarrow \mathbb{N} \) is \( X \)-limitwise monotonic if there exists an \( X \)-computable function \( g : A \times \mathbb{N} \rightarrow \mathbb{N} \) such that for any \( x \in A \)

1. \( g(x, s) \leq g(x, s + 1) \) for all \( s \in \mathbb{N} \),
2. \( f(x) = \lim_{s \rightarrow +\infty} g(x, s) \) (hence the limit is always finite).

THEOREM 4 (A. Frolov, M. Zubkov [8])
Suppose that \( L \) is an \( \eta \)-like linear ordering; then the following are equivalent.

1. \( (|L|, \prec_L, S_L, F_L) \) is a \( 0' \)-computably presentable,
2. \( (|L|, \prec_L, F_L) \) is an \( 0' \)-computably presentable,
3. \( L \cong \sum_{q \in Q} (1 + f_1(q)) \), where the function \( f : Q \rightarrow \mathbb{N} \) is \( 0' \)-limitwise monotonic,
4. \( L \) has a computable copy with \( \Pi^0_1 \) block relation.

COROLLARY 1 (A. Frolov [7])
Every low strongly \( \eta \)-like linear ordering has a computable copy (with \( \Pi^0_1 \) block relation).

PROOF. It is not hard to see that the block relation of a strongly \( \eta \)-like linear ordering is computable relatively to the order relation and the successor relation. Therefore, if a strongly \( \eta \)-like linear ordering \( L \) has a low degree then \( (|L|, \prec_L, S_L, F_L) \) has an \( 0' \)-computable copy. Hence, by Theorem 4, the ordering \( L \) has a computable copy with \( \Pi^0_1 \) block relation. ■

Since any weakly \( \eta \)-like linear ordering contains finite and infinite blocks, we need to extend the previous definition.

DEFINITION 9
A function \( f : A \rightarrow \mathbb{N} \cup \{\omega\} \) is called \( X \)-limitwise monotonic if there exists an \( X \)-computable function \( g : A \times \mathbb{N} \rightarrow \mathbb{N} \) such that for any \( x \in A \)

1. \( g(x, s) \leq g(x, s + 1) \) for all \( s \in \mathbb{N} \),
2. \( f(x) = \lim_{s \rightarrow +\infty} g(x, s) \) if this limit is finite, and \( f(x) = \omega \) otherwise.

THEOREM 5 (A. Frolov, M. Zubkov (unpublished data))
Suppose \( L \) is a weakly \( \eta \)-like linear ordering; then the following are equivalent.

1. \( (|L|, \prec_L, S_L, F_L) \) is an \( 0' \)-computably presentable,
2. \( (|L|, \prec_L, F_L) \) is an \( 0' \)-computably presentable,
3. \( L \cong \sum_{q \in Q} (1 + f_1(q)) \), where functions \( f_1, f_2 : Q \rightarrow \mathbb{N} \cup \{\omega\} \) are \( 0' \)-limitwise monotonic,
4. \( L \) has a computable copy with a \( \Pi^0_1 \) block relation.

PROOF. The theorem is proved by natural changes of the proof of Theorem 4 (see [8]). ■

MAIN THEOREM 1
Let \( L \) be an \( X \)-computable weakly \( \eta \)-like linear ordering with \( X \)-computable successor relation \( S_L \). If \( L \) does not contain strongly \( \eta \)-like interval then \( L \cong \sum_{q \in Q} (1 + f_1(q)) \), where \( f_1, f_2 : Q \rightarrow \mathbb{N} \cup \{\omega\} \) are \( X \)-limitwise monotonic functions.

1Asher M. Kach and Joseph S. Miller have independently announced this result.
Corollary 2

Any low weakly $\eta$-like linear ordering without infinite strongly $\eta$-like interval has a computable copy (with $\Pi^0_1$ block relation).

Proof. Since a linear ordering $L$ has a low degree, the order relation $<_L$ and the successor relation $S_L$ both are $\emptyset'$-computable. From Theorem 1 it follows that there are $\emptyset'$-limitwise monotonic functions $f_1, f_2 : \mathbb{Q} \to \mathbb{N} \cup \{\omega\}$ such that $L \cong \sum_{q \in \mathbb{Q}} (f_1(q))^* + f_2(q))$. By Theorem 2, $L$ has a computable copy with $\Pi^0_1$ block relation.

Proof of Theorem 1. Suppose that $X = \emptyset$ as usual. In other words, let $L$ be a computable linear ordering with computable $S_L$ such that $L/\sim \cong \eta$. Clearly, $B = \{b_0 < b_1 < \cdots\}$ is a $\Pi^0_1$-set, where $[b_i]_L \neq [b_j]_L$ for $i \neq j$, $b_i$ is the least natural number of the class $[b_i]_L$, and for each $x$ there is an $i$ such that $x \in [b_i]_L$. Let $B_s = \{b_{s_0} < b_{s_1} < \cdots\}$ be some $\Pi^0_1$-approximation of $B$. Define $[x]_{s_1} = \{y \in [x]_L \mid y <_L x\}$ and $[x]_{s_2} = \{y \in [x]_L \mid y >_L x\}$.

To define functions $f_1$ and $f_2$ we build their approximations $g_1$ and $g_2$, accordingly, such that $f_i(q)_s = \lim_{s \to +\infty} g_i(q, s)$ for all $q \in \mathbb{Q}$ and $1 \leq i \leq 2$. In our construction, we mark rational numbers by natural numbers. If some rational number $q$ is marked at stage $s$ then either $q$ does not change its mark, or $q$ changes its mark, or $q$ becomes unmarked at a stage $s' > s$. $q_s$ denotes a rational number which is marked by $x$. Fix some computable bijection $N : \mathbb{Q} \to \mathbb{N}$.

Construction.

Stage $s = 0$. Suppose for convenience that $-\infty$ and $+\infty$ are rational numbers and they are marked by $-\infty$ and $+\infty$, accordingly. Moreover, $-\infty <_L x <_L +\infty$ for any element $x$; $b_{-2} = b_{-1}' = -\infty$, $b_{-1} = b_{-1}' = +\infty$ for all $s'$. The all remaining rational numbers at this stage are unmarked.

Stage $s + 1$. Choose the greatest natural number $k \leq s + 1$ such that

$B_k \cap [k \in B_{s+1} \mid k = B_{s+1} \cap [0, \ldots, k) = \{b_{n_0}^1 < \cdots < b_{n_k}^2\}$,

for any $x \in B_{s+1} \mid k$, there is a rational number which is marked by $x$.

For each $x \notin B_{s+1} \mid k$ remove the mark $x$ and consider the following cases.

Case 1

Assume that there exists an unmarked rational number $q$ such that functions $g_1(q, s)$ and $g_2(q, s)$ are defined. Without lost of generality, suppose that such $q$ is the unique. Choose rational numbers $q_s$ and $q_y$ such that

(a) $q_s <_Q q <_Q q_y$;
(b) there is no marked rational number between $q_x$ and $q_y$.

Since $L/\sim \cong \eta$ and $L$ does not contain strongly $\eta$-like interval, one of the following cases always happens: either there is a stage $s' > s$ such that $x \in [y]_L \cap [0, \ldots, s']$ (in this case we do nothing); or there are $z_1, z_2 \in B_{s+1}, t \in B_{s+1}$, and $s'' \geq s$ such that

(a) $x <_L z_1 <_L z_2 <_L y$;
(b) $z_1 <_L t <_L z_2$.
Assume that

\[ L \]

such that

\[ \text{Lemma 1} \]

Define

\[ S \]

\[ \text{Lemma 2} \]

Suppose that the second case happens. Let

\[ \text{Case 2} \]

Proof

\[ \text{Case 3} \]

This completes the construction.

**Lemma 1**

Let \( q_s \) denote the rational number which is marked by \( x \) at stage \( s \). Then for any \( x \in B \) there exists the least stage \( s_x \) such that

1. \( q_s(s) = q_x(s_x) \) for all \( s \geq s_x \),
2. the function \( x \mapsto q_x(s_x) \) is bijection from \( B \) to \( \mathbb{Q} \),
3. \( x < L y \Leftrightarrow q_x(s_x) < q_y(s_y) \) for all \( x, y \in B \).

**Proof.** Let \( s_x \) be the least natural number such that \( s_x \geq s_x \) and \( B_{s_x} \subseteq \{ x \in B \mid x = B_{s_x} \} \) for all \( x' < x \) and all \( s \geq s_x \). By induction, from construction it follows that \( q_x(s) = q_x(s_x) \) for all \( s \geq s_x \); \( q_x(s_x) \neq q_y(s_x') \) and \( x' < y' \Leftrightarrow q_x(s_x) < q_y(s_y') \) for all \( x', y' \in B \) with \( x' < y' \).

It remains to show that for each \( q \in Q \) there is an \( x \in B \) such that \( q = q_x(s_x) \). By induction, let \( q \) be such that for any \( q' \) with \( N(q') < N(q) \) there is an \( x' \) with \( q_x(s_x') = q' \). Let \( s_1 \) be the greatest of all these \( s_x' \). By Case 2, there exists a stage \( s_2 \geq s_1 \) such that functions \( g_1(q, s_2) \) and \( g_2(q, s_2) \) are defined and \( q \) has some mark at this stage.

Choose \( x', y' \in B \) such that \( x' < y' \), \( N(q_x(s_x)) = N(q_y(s_y)) < N(q) \), and \( q_x(s_x) < q y \Leftrightarrow q_x(s_x) < q y \). We have \( x', y' \in B \), \( x' \neq y' \), and \( L \prec \approx \eta \). It follows that there are \( z_1, z_2 \in B \) such that \( x' < L z_1 \prec L z_2 \prec L y' \) (fix the least such \( z_1, z_2 \)). Let \( z \) be the greatest natural number of \( z_1 \) and \( z_2 \). Find a stage \( s_3 \geq s_2 \) such that \( B_z \mid z = B_{s_3} \mid z = B \mid z \) for all \( s \geq s_3 \). By Case 1, after stage \( s_3 \) the rational number \( q \) does not change its mark or changes its mark only once. In other words, there exists a stage \( s_4 \geq s_3 \) such that \( q_x(s) = q_x(s_4) = q \) for some \( x \) and for any \( s \geq s_4 \).

**Lemma 2**

For any \( q \in Q \), \( s \in \mathbb{N} \), and \( 1 \leq i \leq 2 \), \( g_i(q, s) \) is defined, and \( g_i(q, s) \leq g_i(q, s + 1) \).

**Proof.** The lemma immediately follows from previous lemma and the construction.

**Lemma 3**

\[ L = \sum_{q \in Q} (f_1(q))^i + 1 + f_2(q) \]

where \( f_1 : \mathbb{Q} \to \mathbb{N} \cup \{0\}, f_2(q) = \lim_{s \to +\infty} g_i(q, s) \) if the limit is finite, and \( f_2(q) = \omega \) otherwise (\( q \in \mathbb{Q} \), \( 1 \leq i \leq 2 \)).
Theorem 6

Then the linear ordering \( L \) is \( \eta \)-like without strongly \( \eta \)-like interval linear ordering which is not computably presentable.

Proof. If a linear ordering \( L \) does not contain strongly \( \eta \)-like interval then \( Q_L(n, x, y) = 1 \iff [x, y]_L = n \) and hence \( \deg_T(Q_L) \leq \deg_T(<) \lor \deg_T(S_L) \). By Theorem 4 it is enough to construct a \( \mathcal{O} \)-computable structure \((\mathbb{N}, <, L, S_L, P^+_L, P^-_L)\), where \( L = (\mathbb{N}, <) \) is an \( \eta \)-like without strongly \( \eta \)-like interval linear ordering with no computable copy.


Fact 1 for a computable linear ordering \( \sum_{q \in Q} f(q) \), where \( f : Q \to \mathbb{N} \setminus \{0\} \), there exists a \( \mathcal{O} \)-limitwise monotonic function \( g \) whose range is equal to the range of \( f \);

Fact 2 there exists a \( \mathcal{O} \)-computable set \( S \) which is not a range of a \( \mathcal{O} \)-limitwise monotonic function.

3 Low weakly \( \eta \)-like linear orderings

Recently, the author [4] (independently, A. Montalban [3]) proved the following theorem which is helpful to construct low linear orderings (\( \mathcal{O} \)).

Theorem 6

Let \( L = (\mathbb{N}, <, L) \) be a linear ordering such that the structure \((\mathbb{N}, L, S_L)\) is \( X' \)-computable. Then the order \( L \) is \( X' \)-isomorphic to a \( Y \)-computable one for some \( Y \) with \( Y' \leq_T X' \).

Therefore, every linear ordering \( L \) such that \((\mathbb{N}, L, S_L)\) is \( \mathcal{O} \)-computable has a computable copy.

To construct low 2 linear orderings, we need the following theorem.

Theorem 7

Let \( L = (\mathbb{N}, <, L) \) be a linear ordering such that the structure \((\mathbb{N}, L, S_L, Q_L, P^+_L, P^-_L)\) is \( X' \)-computable. Then the structure \((\mathbb{N}, L, S_L, Q_L, P^+_L, P^-_L)\) is \( X' \)-computable.

Proof. It follows from the construction that \( g_1(q_L(s_x), s) = [x]_{<L} \cap \{0, \ldots, s\} \) and \( g_2(q_L(s_x), s) = [x]_{>L} \cap \{0, \ldots, s\} \) for all \( s \geq s_x \) and \( x \in B \). Therefore, \( \lim_{s \to +\infty} g_1(q_L(s_x), s) = [x]_{<L} \) and \( \lim_{s \to +\infty} g_2(q_L(s_x), s) = [x]_{>L} \). All conditions of the lemma immediately follow from this.

\[\blacksquare\]

\[\blacksquare\]
Fix the set $S$ from Fact 2 (assuming that $0, 1 \notin S$). We construct $L$ such that $S = \{n\}$ there is a block whose power is $n$. By Fact 1, $L$ has no a computable presentation. Fix the oracle $\emptyset'$.

**Construction**

**Stage $s=0$**. Let the relations $<_L, S_L, P^+_L$, and $P^-_L$ are undefined.

**Stage $s+1$**. Suppose that at stage $s$ the relations $<_L, S_L, P^+_L$, and $P^-_L$ are defined on $\{y \mid y < k\}$. Let

$$x_1 <_L y_1 <_L x_2 <_L y_2 <_L \cdots <_L x_m <_L y_m,$$

where $x_1, x_2, \ldots, x_m$ are all elements $x$ such that $P^+_L(x)$; $y_1, y_2, \ldots, y_m$ are all elements $y$ such that $P^-_L(y)$.

Let $z \in S$ be the least natural number such that $z \notin \{n \mid (\exists 1 \leq i \leq m([x_i, y_i]_L) = n\})$. We put new $z$ elements $t_1, \ldots, t_z$ immediately left of each $x_i$ and new $z$ elements $t_{z+1}, \ldots, t_{2z}$ immediately right of each $y_j$. Define for any $j \in [1, z+1]$

$$S_L(t_j, t_{j+1}), \ldots, S_L(t_{z+j-2}, t_{z+j-1}),$$

$$\neg S_L(t_1, x_i) \text{ and } \neg S_L(t_2, x_i) \text{ for any } x_i \notin \{t_1, \ldots, t_{z+j-1}\} \text{ and } j \leq i \leq z+j-1,$$

$$P^+_L(t_j), \neg P^-_L(t_{j+1}), \ldots, \neg P^-_L(t_{z+j-1}),$$

$$\neg P^+_L(t_{z+j-2}), \ldots, \neg P^+_L(t_{z+j-1}).$$

This completes the construction.

From construction immediately it follows that the linear ordering $L$ is $\eta$-like without strongly $\eta$-like interval. By comments above, $L$ has a low$_2$ copy and has no a computable presentation. \hfill \blacksquare

**Proof of Theorem** Suppose that $X = \emptyset$ as usual. In other words, let $L = (\mathbb{N}, <_L)$ be a linear ordering such that the structure $(\mathbb{N}, <_L, S_L, P^+_L, P^-_L, Q_L)$ is $\emptyset'$-computable.

$\sigma$ always denotes a finite structure $(|\sigma|, <_\sigma, S_\sigma)$, where

1. $(|\sigma|, <_\sigma)$ is a finite linear ordering,
2. for any $x, y, S_\sigma(x, y)$ implies $(\forall z \in |\sigma|)\neg(x <_\sigma z <_\sigma y)$.

We say that $f: |\sigma| \to |L|$ is a correct embedding from $\sigma$ into $L$ if for any $x, y \in |\sigma|$

1. $x <_\sigma y \iff f(x) <_L f(y),$
2. $S_\sigma(x, y) \iff S_L(f(x), f(y)).$

Fix the oracle $\emptyset$.

**Construction**

**Stage $s=0$** Let $\sigma_0$ be an empty structure.

**Stage $s+1$** Suppose that we have a structure $\sigma_s = (|\sigma_s|, <_{\sigma_s}, S_{\sigma_s})$ and a correct embedding $f_s$ from $\sigma_s$ into $L$ such that any natural number $<_s$ is contained in both $|\sigma_s|$ and $f(|\sigma_s|)$.

**Substage $1$** Let $u$ be the least natural number such that $u \notin |\sigma_s|$, $t$ be the least natural number such that $t \notin f(|\sigma_s|)$. Define $\sigma'_s = (|\sigma_s| \cup [u], <_{\sigma'_s}, S_{\sigma'_s}), f'_s(u) = t$, and for any $x \in |\sigma'_s|$,

1. $f'_s(x) = f_s(x),$
2. $u <_{\sigma'_s} x \iff t <_L f_s(x),$
3. $S_{\sigma'_s}(u, x) \iff S_L(t, f_s(x))$ and $S_{\sigma'_s}(x, u) \iff S_L(f_s(x), t)$.
Note that \( s \in \sigma_i' \), \( s \in f'_i(\sigma_i') \), and \( f'_i \) is a correct embedding \( \sigma_i' \) into \( L \).

Substage 2 By induction, we will define sequences \( \sigma^0 \subseteq \sigma^1 \subseteq \cdots \subseteq \sigma^k \subseteq \cdots \) such that \( f^k \) is a correct embedding from \( \sigma^k \) into \( L \), \( Q^{\sigma^k} \) is an additional predicate on \( \mathbb{N} \times |\sigma^k| \times |\sigma^k| \). Let \( \sigma^0 = \sigma_i', f^0 = f'_i \), \( Q^{\sigma^0}(n,x,y) = 0 \) for any \( n \in \mathbb{N}, x,y \in |\sigma^0| \). Suppose that \( \sigma^k, f^k, \) and \( Q^{\sigma^k} \) are defined such that \( f^k \) is a correct embedding from \( \sigma^k \) into \( L \), there are only finitely many \( n \) such that \( Q^{\sigma^k}(n,x,y) = 1 \) for any \( x,y \in |\sigma^k| \) (moreover, the set \( Q(x,y) = |n|Q^{\sigma^k}(n,x,y) = 1 \) is uniformly computable).

It is not hard to see that the following formula is \( \emptyset' \)-computable

\[
\Gamma(\sigma^k) = (\exists \sigma' \supseteq \sigma^k)(\exists \sigma')(\forall x,y \in |\sigma^k|)(\forall \gamma) \neg \sigma' \Gamma_{\gamma}(x,y),
\]

(1) \( \psi_{\sigma',\gamma}(x) \).

(2) \( P_{\lambda}^{\sigma'}(f^k(x)) \rightarrow \neg \sigma' \Gamma_{\eta}(x,x') \).

(3) \( P_{\lambda}^{\sigma'}(f^k(x)) \rightarrow \neg \sigma' \Gamma_{\eta}(x',y) \).

(4) \( (x <_{\sigma'} y \& Q^{\sigma^k}(n,x,y)) \rightarrow (\exists x', \ldots, x'_{n}) (\forall 0 \leq i < n)(x <_{\sigma_i'} x'_{i} <_{\sigma_i'} \cdots <_{\sigma_i'} x'_{n} <_{\sigma_i'} y) \& S_{\sigma_i'}(x', x'_{n+1}) \).

If \( \Gamma(\sigma^k) \) fails then define \( \sigma_{i+1} = \sigma^k \) and \( f_{i+1} = f^k \). This completes the stage \( s+1 \).

Suppose that \( \Gamma(\sigma^k) \) holds and \( \sigma' \supseteq \sigma^k \) satisfies conditions of the formulae. Without lost of generality, suppose that only one of the following cases happens.

Case 1

Let \( |\sigma'| = |\sigma^k| \cup \{x'_1, \ldots, x'_j\} \), where \( x'_1, \ldots, x'_j \notin |\sigma^k| \), \( x'_0 <_{\sigma'} x'_1 <_{\sigma'} \cdots <_{\sigma'} x'_j \), and \( S_{\sigma'}(x'_0, x'_1), \ldots, S_{\sigma'}(x'_{j-1}, x'_j) \) for some \( x'_0 \in |\sigma^k| \). We always find

(1) either a correct embedding \( f' \supseteq f^k \) from \( \sigma' \) into \( L \);

Define \( \sigma_{i+1} = \sigma' \) and \( f_{i+1} = f' \). This completes stage \( s+1 \).

(2) or elements \( x_0, x_1, \ldots, x_j \) such that \( x_0 = f^k(x'_0) <_{L} x_1 <_{L} \cdots <_{L} x_j \), \( S_L(x_0, x_1), \ldots, S_L(x_{j-1}, x_j) \), \( j < i \), and \( P_{\lambda}^{\sigma^k}(x_j) \) or \( x_j \not\in |\sigma^k| \).

Define \( \sigma'_{i+1} = (|\sigma^k| \cup \{x'_1, \ldots, x'_j\}, <_{\sigma'} \sigma_{i+1}^\prime) \), \( f^k_{i+1}(x'_i) = x_u \) for \( 1 \leq u \leq j \), \( f^k_{i+1}(x) = f^k(x) \) for any \( x \in |\sigma_L| \), \( Q^{\sigma^k}(n,x,y) = Q^{\sigma^k}(n,x',y') = 0 \) if either \( x' \notin |\sigma^k| \) or \( y' \not\in |\sigma^k| \) (for any \( n \in \mathbb{N} \)).

Case 2

Let \( |\sigma'| = |\sigma^k| \cup \{x'_1, \ldots, x'_j\} \), where \( x'_1, \ldots, x'_j \notin |\sigma^k| \), \( x'_1 <_{\sigma'} \cdots <_{\sigma'} x'_j <_{\sigma'} x'_{i+1} \) \& \( S_{\sigma'}(x'_j, x'_{j+1}) \) for some \( x'_{j+1} \in |\sigma^k| \). This case is considered similarly to Case 1.

Case 3

Let \( |\sigma'| = |\sigma^k| \cup \{x'_1, \ldots, x'_j\} \), where for some \( x'_0, x'_{i+1} \in |\sigma^k| \),

\[
\begin{align*}
x'_1, \ldots, x'_j &\not\in |\sigma^k|, \\
x'_0 <_{\sigma'} x'_1 <_{\sigma'} \cdots <_{\sigma'} x'_j <_{\sigma'} x'_{i+1}, \\
\neg S_{\sigma'}(x'_0, x'_1), &\neg S_{\sigma'}(x'_j, x'_{i+1}), \\
S_{\sigma'}(x'_1, x'_2), &\ldots, S_{\sigma'}(x'_{j-1}, x'_j).
\end{align*}
\]

Necessarily, either \( Q_{L}(i, f^k(x'_0), f^k(x'_{j+1})) \) or we find a correct embedding \( f' \supseteq f^k \) from \( \sigma' \) into \( L \).

\(^2\text{Since } \sigma \text{ is finite, all } \forall \text{-quantifiers are bounded.}\)
(1) In the first case we define $\sigma^{k+1} = \sigma^k$, $f^{k+1} = f^k$, $Q^{k+1}(i, x_0, x_{i+1}) = 1$, and $Q^{k+1}(n, x, y) = Q^k(n, x, y)$ for any $x, y \in [\sigma^k] - [x_0, x_{i+1}]$, and $n \in \mathbb{N}$.

(2) In the second case we define $\sigma_{k+1} = \sigma^r$ and $f_{k+1} = f^r$. This completes stage $s + 1$.

It is not hard to see that the sequence $[\sigma^k]$ is finite. In other words, after finitely many steps the structure $\sigma_{s+1} \supseteq \sigma_s$ and the correct embedding $f_{s+1} \supseteq f_s$ from $\sigma_{s+1}$ into $L$ are defined.

This completes the construction.

It is obvious that $\sigma_s \subseteq \sigma_{s+1}$. Since $s \in [\sigma_{s+1}]$, we have $\bigcup_{s \in \mathbb{N}} [\sigma_s] = \mathbb{N}$. Let $R = (\mathbb{N}, <_R)$, where $<_R = \bigcup_{s \in \mathbb{N}} <_{\sigma_s}$. It follows from the construction that $S_R = \bigcup_{s \in \mathbb{N}} S_{\sigma_s}$.

Let $f = \lim_{s \to \infty} f_s$. We have that $f_s$ is a correct embedding $\sigma_s$ into $L$ and $s \in f_{s+1}([\sigma_{s+1}])$. It follows that $f$ is an isomorphism from $R$ to $L$.

Since $\psi_{\langle \mathbb{N}, <_R, S_R \rangle}(e) \downarrow \iff \psi_{\sigma_s}(e) \downarrow$, the predicate $H(e) = \psi_{\langle \mathbb{N}, <_R, S_R \rangle}(e)$ is $\emptyset'$-computable and hence the structure $(\mathbb{N}, <_R, S_R)$ has a low degree.

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References


Remark that $Q^k(i, x_0, x_{i+1}) = 0$ for all $i \leq k$. It follows that Case 3 happens only finitely many times.
Low linear orderings


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