Contents

Vol. 59, No. 11, 2015

Simultaneous English language translation of the journal is available from Allerton Press, Inc.
Distributed worldwide by Springer. *Russian Mathematics (Iz. VUZ)* ISSN 1066-369X.

Kolmogorov-Type Inequalities for Derived Functions of Two Variables with Application for Approximation by an "Angle"
S. B. Vakarchuk and A. V. Shvachko 1

The Listing of Topologies Close to the Discrete One on Finite Sets
I. G. Velichko, P. G. Stegantseva, and N. P. Bashova 19

Comparative Complexity of Quantum and Classical OBDDs for Total and Partial Functions
A. F. Gainutdinova 26

The Dirichlet Problem for an Equation of Mixed Elliptic-Hyperbolic Type with Variable Potential
N. V. Martem'yanova 36

New Weakly Periodic Gibbs Measures of Ising Model on Cayley Tree
M. M. Rakhmatullaev 45

On Minimal Poisson Algebras
S. M. Ratseev 54

Integral Representation of a Solution to Three-Dimensional Elliptic Equation with Strong Degeneration
G. Z. Khabibullina, S. V. Makletsov, and R. M. Mavlyaviev 62

Brief communications

On Minimization of Cavity Drag of Hydrofoils
I. R. Kayumov and D. V. Maklakov 68

Noncommutative Vector-Valued Symmetric Hardy Spaces
K. S. Tulenov
(Submitted by A.M. Bikchentaev) 74

ISSN 1066-369X, Russian Mathematics (Iz. VUZ), 2015, Vol. 59, No. 11, pp. 68–73. © Allerton Press, Inc., 2015.
Original Russian Text © I.R. Kayumov, D.V. Maklakov, 2015, published in Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika, 2015, No. 11, pp. 80–96.

On Minimization of Cavity Drag of Hydrofoils

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Received April 1, 2015

Abstract—In this work we investigate the limiting values of the lift and drag coefficients of profiles in the Helmholtz–Kirchhoff (infinite cavity) flow. The coefficients are based on the wetted arc length of profile surfaces. For a given value of the lift coefficient and additional restrictions from above and below on the velocity distribution along the profile surface we find global minimum of the drag coefficient.

DOI: 10.3103/S1066369X15110080

Keywords: *extremal problem, ideal fluid, Helmholtz–Kirchhoff model, cavity flows.*

In this work we generalize the results obtained by the authors in papers [1, 2]. Consider the Helmholtz–Kirchhoff (infinite cavity) flow with incident velocity v_0 over a profile (Figure). We assume that the flow is steady and irrotational and the fluid is ideal and incompressible.

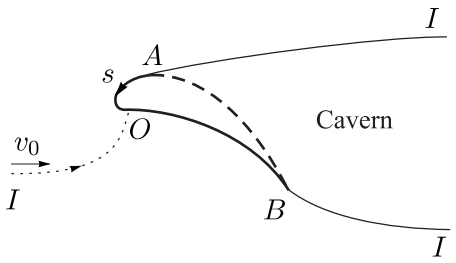

Figure. Profile in the Helmholtz–Kirchhoff flow.

According to the Helmholtz–Kirchhoff model the flow detaches from the profile surface at the points A and B, and an infinite cavity with a constant pressure, equal to the incident pressure, forms behind the profile. The velocity on the free streamlines AI and BI is constant and equals the incident velocity v_0. We denote the stagnation point by O and reckon the arc abscissa s from the point A.

Formulas for the lift and drag force have been recently obtained in [3, 4]:

$$L = \rho v_0 \int_0^l (\mathbf{v} \cdot \boldsymbol{\tau}) \log \frac{v_0}{v}\, ds, \quad D = \frac{\rho v_0}{4\pi} \left(\int_0^l \frac{v}{\sqrt{\varphi}} \log \frac{v_0}{v}\, ds \right)^2, \tag{1}$$

where l is the length of the wetted arc AOB of the profile, $\varphi = \varphi(s)$ is the distribution of potential along AOB:

$$\varphi = \int_s^{l_1} v(s) ds \ \text{ for } \ 0 \le s \le l_1, \quad \varphi = \int_{l_1}^{s} v(s) ds \ \text{ for } \ l_1 \le s \le l,$$

l_1 is the arc abscissa of the critical point O.

Let us introduce the lift and drag coefficients $C_L = 2L/(\rho v_0^2 l)$ and $C_D = 2D/(\rho v_0^2 l)$ based on the wetted arc length l of the profile. In [2] we have solved the problem of finding a velocity distribution

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along the surface of a cavitating profile in such a way that under a given lift force L (lift coefficient C_L) the distribution provides minimum of drag force D (minimum value of C_D). From the point of view of the physical realizability of profiles an interesting question arises: To solve an analogous problem with restriction from below on the velocity on the profile surface.

Let $l_2 = l - l_1$ be the length of the arc OB. We introduce two dimensionless functions $u_1(\sigma)$ and $u_2(\sigma)$, $0 \le \sigma \le 1$, such that

$$\frac{v}{v_0} = \begin{cases} u_1\big(\frac{l_1 - s}{l_1}\big) & \text{on } OA, \\ u_2\big(\frac{s - l_1}{l_2}\big) & \text{on } OB. \end{cases} \tag{2}$$

Since the velocity $v \ge 0$, the functions $u_1(\sigma)$ and $u_2(\sigma)$ are nonnegative. Under the Brillouin condition they satisfy the inequalities

$$u_1(\sigma) \le 1, \quad u_2(\sigma) \le 1. \tag{3}$$

By means of (1) we express the lift and drag coefficients in terms of $u_1(\sigma)$ and $u_2(\sigma)$:

$$C_L = 2\{(1-\varepsilon)I[u_2] - \varepsilon I[u_1]\}, \quad C_D = \frac{1}{2\pi}\big\{\sqrt{1-\varepsilon}\, J[u_2] + \sqrt{\varepsilon}\, J[u_1]\big\}^2, \tag{4}$$

where $\varepsilon = l_1/l$, $I[u]$ and $J[u]$ are nonlinear functionals of $u(\sigma)$, $0 \le \sigma \le 1$:

$$I[u] = -\int_0^1 u(\sigma)\log u(\sigma)\, d\sigma, \quad J[u] = -\int_0^1 \frac{u(\sigma)\log u(\sigma)\, d\sigma}{\sqrt{\int_0^\sigma u(\sigma_1) d\sigma_1}}. \tag{5}$$

As one can see from (5), under the Brillouin condition (3) the values of the functionals $I[u]$ and $J[u]$ at $u = u_1(\sigma)$ and $u = u_2(\sigma)$ are nonnegative.

Let us rewrite $I[u]$ and $J[u]$ in terms of classical functionals of calculus of variations. To do so we transform the function $u(\sigma)$ to $\lambda(\sigma)$:

$$\lambda(\sigma) = \sqrt{2\int_0^\sigma u(\sigma_1) d\sigma_1}.$$

Then

$$I[\lambda] = -\int_0^1 \lambda\lambda' \log(\lambda\lambda') d\sigma, \quad J[\lambda] = -\sqrt{2}\int_0^1 \lambda' \log(\lambda\lambda') d\sigma.$$

It is clear that $\lambda(\sigma) \ge 0$. Besides, $u(\sigma) = \lambda(\sigma)\lambda'(\sigma)$, hence $\lambda'(\sigma) \ge 0$.

The solution of the basic problem of finding absolute minimum of C_D under a given value of C_L is based on solving the following auxiliary variational problem.

Auxiliary problem. Find the function $\lambda(\sigma)$, $\sigma \in [0,1]$:

$$\lambda(0) = 0, \quad \lambda'(\sigma) \ge 0,$$

which delivers an absolute minimum to the functional $J[\lambda]$ under the condition $I[\lambda] = q$ (q is given), and the complementary constraint

$$\frac{1}{d} \le \lambda(\sigma)\lambda'(\sigma) \le 1. \tag{6}$$

Without the complementary constraint from below in (6) this problem has been solved in [1] (see also [2]). To obtain the solution we have used the technique developed earlier in papers [5–9] to investigate extremal problems of the jet and cavitation theory. The technique is based on Jensen's inequality. The complementary restriction from below makes the problem more complex and in this case the technique mentioned above fails to be generalized.

We denote by $J_{\min}(q)$ the global minimum of the functional $J[\lambda]$ for a given value of $I[\lambda] = q$. A solution to the auxiliary variational problem has been obtained by means of an algorithm which can be described as follows.

Step 1. By reconstructing the integrands we prove that the variational problem can be solved in the class of functions such that $\lambda\lambda'$ does not decrease on the segment $[0, 1]$.

Step 2. By making use of Step 1 and the classical Helly selection theorem for convergence of functions of bounded variation we prove the existence theorem for the auxiliary problem.

Step 3. We prove that $\lambda\lambda'$ is continuous, moreover, it is constant on two intervals and on the remainder mean interval we find its analytical representation by the integral of the Euler–Lagrange equation.

Let us proceed now to the realization of this algorithm.

Lemma 1. *Let the function λ be such that $\lambda(0) = 0$, $\lambda\lambda'(\sigma)$ be measurable on $[0, 1]$ and satisfies the inequalities*

$$\frac{1}{e} \le \frac{1}{d} \le \lambda\lambda'(\sigma) \le 1.$$

Then there exists a monotonic function h such that $h(0) = 0$, hh' does not increase on $[0, 1]$ and satisfies the conditions

$$\int_0^1 \lambda'\lambda \ln(\lambda\lambda')\mathrm{d}\sigma = \int_0^1 hh' \ln(hh')\mathrm{d}\sigma,$$
$$\int_0^1 \lambda' \ln(\lambda\lambda')\mathrm{d}\sigma \le \int_0^1 h' \ln(hh')\mathrm{d}\sigma,$$
$$\frac{1}{d} \le \lambda\lambda'(\sigma) \le 1.$$

Lemma 2. *Let $k < 0$. In the class of functions satisfying the conditions*

$$\lambda(0) = 0, \ \ \frac{1}{d} \le \lambda(\sigma)\lambda'(\sigma) \le 1 \tag{7}$$

there exists a solution to the extremal problem

$$P[\lambda] = -\int_0^1 \lambda'(\lambda + k) \log(\lambda\lambda')d\sigma \to \max$$

and for this solution the function $\lambda\lambda'(\sigma)$ does not increase on the segment $[0, 1]$.

Proof. In [2] it has been proved that the functional $P[\lambda]$ is bounded in a wider class of functions which have no constraint from below on $\lambda\lambda'$. This means that there exists $\sup P[\lambda] = S$ in the class of functions satisfying (7). Because of this, there exists a sequence of functions g_n satisfying (7) such that

$$\lim_{n\to\infty} P[g_n] = S.$$

We reconstruct each of the functions g_n in accordance with Lemma 1. As a result we get the sequence of functions h_n satisfying (7), moreover $P[h_n] \ge P[g_n]$. Since $h_n h_n'$ is monotonic, the function $h_n h_n'$ has a bounded variation. Hence, by the Helly selection theorem ([10], P. 422) it is possible to extract from the sequence $h_n h_n'$ an everywhere convergent subsequence. The limit of this subsequence we denote by $\lambda\lambda'$. It is evident that $\lambda\lambda'$ does not increase. Further, by Lebesgue's dominated convergence theorem

$$P[\lambda] = -\int_0^1 \lambda'(\lambda + k) \ln(\lambda\lambda')\mathrm{d}\sigma = -\lim_{n\to\infty} \int_0^1 h_n'(h_n + k) \ln(h_n h_n')\mathrm{d}\sigma \ge S.$$

Therefore, $P[\lambda] = S$. Conditions (7) for λ are fulfilled by its construction. □

Lemma 3. *Let $k<0$. Then in the class of functions satisfying conditions* (7) *a solution to the extremal problem*

$$P[\lambda]=-\int_0^1 \lambda'(\lambda+k)\log(\lambda\lambda')d\sigma\to\max$$

can be sought in the class of continuous piecewise differentiable functions satisfying the conditions

$$\lambda(\sigma)=\sqrt{2\sigma},\ \ \sigma\in[0,\gamma_1],\ \ \lambda'(\sigma)(\lambda(\sigma)+k)=c,\ \ \sigma\in[\gamma_1,\gamma_2],\ \ \lambda'(\sigma)\lambda(\sigma)=1/d,\ \ \sigma\in[\gamma_2,1].$$

Theorem 1. *The function $J_{\max}(q)$ is defined by the parametric equations*

$$2qd=k^2cd\frac{d-1}{(1-c)(cd-1)}+\left(2-dk^2+\frac{k^2(d-1)}{(c-1)(cd-1)}\right)\ln d-k^2\ln\frac{c-1}{cd-1},\tag{8}$$

$$J_{\min}(q)=\frac{2\sqrt{2}}{k}(c-q),\tag{9}$$

where

$$k=k(c)=-(cd-\ln d)\sqrt{\frac{2(c-1)(cd-1)}{c^2d^2(d-1)-2cd(d-1)\ln d+(cd(1+d)-1-c^2d^2)\ln^2 d}},$$

and $c\in[0,\ln d/d]$.

Proof. Consider the functional

$$I[\lambda]+\frac{k}{\sqrt{2}}J[\lambda]=q+\frac{k}{\sqrt{2}}J[\lambda].$$

At a fixed negative value of k, as has been mentioned above, the functional has a unique point of global maximum to which corresponds a unique value of the parameter c. Therefore, at the same c we have a global minimum of the functional J at a fixed q.

Let $a=\frac{k}{c-1}$, $b=\frac{k}{cd-1}$. Direct calculations demonstrate that

$$q=-\int_0^1\lambda\lambda'\log(\lambda\lambda')d\sigma=\frac{1}{2d}\left[k(b-a)d-k^2\ln\frac{b}{a}+(2+ab(d-1)-k^2d)\ln d\right],$$

$$\frac{J_{\min}(q)}{\sqrt{2}}=k\ln\frac{b}{a}+\left(k+\sqrt{\frac{2}{d}+\frac{ab(d-1)}{d}}\right)\ln d,$$

whence, after simple transformations we get (9). □

We emphasize once again that Theorem 1 is a nontrivial generalization of theorem 1 from [2]. An important difference is that in [2] we have no constraint from below in (7) and the parameterization of solution has been carried out by the parameter $b=\lambda(1)$. In the new problem the solution is parameterized by the parameter c, and in the particular case $d=e$ the reparameterization is possible in terms of b.

To solve the basic problem we prove

Lemma 4. *The function $J_{\min}(q)$ strictly increases on the segment $[0,q_{\max}]$.*

Proof. Using (8), (9), we obtain the equation

$$\frac{dJ_{\min}(q)}{dq}=\frac{dJ_{\min}(q)}{dc}\Big/\frac{dq}{dc}=-\frac{\sqrt{2}}{k}.$$

But for the function $J_{\min}(q)$ the parameter k is always negative (see Theorem 1). □

Consider now the first equation in (4). Taking into account that $\varepsilon = l_1/l \in [0,1]$, we conclude that

$$C_L = 2\{(1-\varepsilon)I[u_2] - \varepsilon I[u_1]\} \le 2I[u_2].$$

Since the maximum value of the functional $I[u] = q_{\max} = \log d/d$, the lift coefficient C_L belongs to $(0, 2\log g/d]$.

Theorem 2. *At a given value of the lift coefficient $C_L \in (0, 2\log d/d]$ the global minimum of the drag coefficient $C_{D\min} = \frac{1}{2\pi} J^2_{\min}(C_L/2)$.*

Proof. Let us set $q_1 = I[u_1]$, $q_2 = I[u_2]$. It is clear that

$$\varepsilon \in [0,1), \quad 0 \le q_1, q_2 \le q_{\max}, \quad (1-\varepsilon)q_2 - \varepsilon q_1 = C_L/2 \in (0, q_{\max}], \tag{10}$$

$$C_{D\min} = \min_{\varepsilon, q_1, q_2} \frac{1}{2\pi}\{\sqrt{1-\varepsilon}\, J_{\min}(q_2) + \sqrt{\varepsilon}\, J_{\min}(q_1)\}^2.$$

It should be noted that $\varepsilon \ne 1$, because for $\varepsilon = 1$ constraints (10) become contradictory. Expressing q_2 in terms of q_1 by means of the last relation in (10), we get

$$C_{D\min} = \min_{\varepsilon, q_1} \left\{ \sqrt{1-\varepsilon}\, J_{\min}\left(\frac{C_L/2}{1-\varepsilon} + \frac{\varepsilon q_1}{1-\varepsilon}\right) + \sqrt{\varepsilon}\, J_{\min}(q_1) \right\}^2.$$

By Lemma 4 the function to be minimized strictly increases with respect to q_1. Hence, the minimum is attained at the point $q_1 = 0$, i.e.,

$$C_{D\min} = \min_{\varepsilon} \frac{1}{2\pi} \left\{ \sqrt{1-\varepsilon}\, J_{\min}\left(\frac{C_L/2}{1-\varepsilon}\right) \right\}^2. \tag{11}$$

To prove the theorem we need to establish that the minimum is attained at the point $\varepsilon = 0$. This will be so if the function to be minimized in (11) strictly increases with respect to ε. Let

$$q = \frac{C_L/2}{1-\varepsilon}$$

and consider the function $J^2_{\min}(q)/q$. It is easy to see that this function strictly increases. Indeed, calculating its derivative, we infer that the strict increase of $J^2_{\min}(q)/q$ is equivalent to the inequality

$$qJ'_{\min}(q) - \frac{J_{\min}(q)}{2} > 0.$$

Using Lemma 4, we deduce after simple transformations that

$$qJ'_{\min}(q) - \frac{J_{\min}(q)}{2} = -\frac{\sqrt{2}c}{k}.$$

The latter expression is positive because $k < 0$ and $c > 0$ (see Theorem 1). The strict increase of $J^2_{\min}(q)/q$ leads to the strict increase of the function to be minimized in (11). □

ACKNOWLEDGMENTS

Supported by the Russian Foundation for Basic Research (RBFR, project No. 15-01-06029-a) and by RFBR and the Government of the Republic of Tatarstan (project No. 15-41-02433).

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Translated by D. V. Maklakov

ISSN 1066-369X, Russian Mathematics (Iz. VUZ), 2015, Vol. 59, No. 11, pp. 74–79. © Allerton Press, Inc., 2015.
Original Russian Text © K.S. Tulenov, 2015, published in Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika, 2015, No. 11, pp. 87–93.

Noncommutative Vector-Valued Symmetric Hardy Spaces

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Received March 10, 2015

Abstract—We introduce noncommutative vector-valued symmetric Hardy spaces and demonstrate that they are quasi-Banach ones. For these spaces we prove analogs of some basic properties of the Hardy space.

DOI: 10.3103/S1066369X15110092

Keywords: *von Neumann algebra, subdiagonal algebra, noncommutative vector-valued symmetric Hardy space, conditional expectation.*

1. INTRODUCTION

Let $\mathcal{M}$, a finite von Neumann algebra with an exact normal trace τ act in the Hilbert space $\mathcal{H}$. Let $\mathcal{D}$ be a subalgebra of von Neumann algebra $\mathcal{M}$, $\Phi : \mathcal{M} \to \mathcal{D}$ be a unique normal exact conditional expectation such that $\tau \circ \Phi = \tau$. Then an $*$-weakly closed subalgebra $\mathcal{A}$ of the algebra $\mathcal{M}$ with respect to Φ is a finite subdiagonal algebra [1], if the following conditions are fulfilled:

(i) $\mathcal{A} + \mathcal{A}^*$ is $*$-weakly closed in $\mathcal{M}$;

(ii) Φ is multiplicative on $\mathcal{A}$, i.e., $\Phi(ab) = \Phi(a)\Phi(b) \quad \forall a, b \in \mathcal{A}$;

(iii) $\mathcal{A} \cap \mathcal{A}^* = \mathcal{D}$, where $\mathcal{A}^* = \{a^* : a \in \mathcal{A}\}$.

An algebra $\mathcal{D}$ is said to be a diagonal algebra for $\mathcal{A}$. In [2] it is proved that each finite subdiagonal algebra $\mathcal{A}$ is maximal. This means that if there exists another algebra $\mathcal{B}$ that contains $\mathcal{A}$ with respect to Φ, then $\mathcal{B} = \mathcal{A}$. Let $0 < p \leq \infty$. By $L_p(\mathcal{M})$ let us denote a noncommutative L_p-space associated with $(\mathcal{M}, \tau)$. For $L_\infty(\mathcal{M})$, let us take $\mathcal{M}$ with an operator norm [3]. By $\|\cdot\|_p$ denote the norm of the space $L_p(\mathcal{M})$. For $p < \infty$ we define $H_p(\mathcal{A})$ as a closure of $\mathcal{A}$ with respect to the norm $L_p(\mathcal{M})$, and assume $H_\infty(\mathcal{A}) = \mathcal{A}$ for $p = \infty$. These spaces are also called Hardy spaces associated with $\mathcal{A}$. They are noncommutative analogs of classical Hardy spaces on a torus $\mathbb{T}$. Hardy spaces are in detail covered in surveys [1], [3] and [4]. Some properties of the space $H_p(\mathcal{A})$ were considered in the work of G. Sh. Skvortsova [5]. After the pioneering work by W. B. Arveson [4] a great interest to noncommutative Hardy spaces emerged. For more detail, see [3, 6, 7].

A closed everywhere densely defined in $\mathcal{H}$ operator x with a domain $D(x)$ is called associated with $\mathcal{M}$ ($x\eta\mathcal{M}$), if and only if $ux = xu$ for all unitary u that belong to the commutant $\mathcal{M}'$ of the algebra $\mathcal{M}$. An operator $x\eta\mathcal{M}$ is called τ-measurable if for an arbitrary $\varepsilon > 0$ there exists a projector $e \in \mathcal{M}$ such that $e(\mathcal{H}) \subseteq D(x)$ and $\tau(1 - e) < \varepsilon$. Denote the set of all τ-measurable operators as $L_0(\mathcal{M})$. A set $L_0(\mathcal{M})$ is an $*$-algebra with strong sum and product, that are closures of the algebraic sum and multiplication [3]. It is a well-known fact that for each operator $x\eta\mathcal{M}$ in $\mathcal{H}$, all spectral projectors $e_s^\perp(|x|) = \chi_{(s,\infty)}(|x|)$, corresponding to the interval (s, ∞), belong to $\mathcal{M}$, and $x \in L_0(\mathcal{M})$ if and only if $\tau(e_s^\perp(|x|)) < \infty$ for some $s \in \mathbb{R}$. Define a non-increasing permutation (or generalized singular values) of the operator $x \in L_0(\mathcal{M})$ in the following way:

$$\mu_t(x) = \inf\{s > 0 : \lambda_s(x) \leq t\}, \quad t > 0,$$

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where $\lambda_s(x) = \tau(e_s^{\perp}(|x|)),\ s > 0$. A function $s \mapsto \lambda_s(x)$ is called the distribution function of the operator x. For more detail on generalized singular numbers of τ-measurable operators see [8]. Define a symmetric space of operators $L_E(\mathcal{M})$, constructed with respect to noncommutative measurable space $(\mathcal{M}, \tau)$ and symmetric Banach space of functions. Let (Ω, Σ, μ) be a complete measurable space with a σ-finite measure and $L_0(\Omega)$ be the space of all equivalence classes of μ-measurable real-valued functions defined on Ω. We will call a quasi-Banach space $E = (E, \|\cdot\|_E)$ a quasi-Banach ideal space on Ω if E is a linear subspace of $L_0(\Omega)$ and it satisfies the so-called ideal property. That means that if $y \in E$, $x \in L_0(\Omega)$ and $|x(t)| = |y(t)|$ for μ-almost all $t \in \Omega$, then $x \in E$ and $\|x\|_E \le \|y\|_E$. A symmetric quasi-Banach space on $[0, 1]$ is understood as a quasi-Banach lattice E of measurable functions on $[0, 1]$, satisfying the following conditions: (i) E contains all simple functions; (ii) if $x \in E$ and y is a measurable function such that $|y|$ and $|x|$ have the same distribution, then $y \in E$ and $\|x\|_E \le \|y\|_E$. For convenience, let us assume that E, in addition, satisfies the condition $0 \le x_n \uparrow x, x_n, x \in E \Rightarrow \|x_n\|_E \uparrow \|x\|_E$. We will say that in E a (quasi-)norm is order continuous, if for each lattice $(x_i)_{i\in I}$ in E such that $x_i \downarrow 0$ we have $\|x_i\|_E \downarrow 0$. A space E is called totally symmetric if for $x \in L_0[0, 1]$ and $y \in E$ with $x \prec\prec y$ we obtain that $x \in E$ and $\|x\|_E \le \|y\|_E$; here $x \prec\prec y$ means that x is dominated by element y in the sense of Hardy–Littlewood:

$$\int_0^t \mu_s(x)ds \le \int_0^t \mu_s(y)ds \ \text{ for all } \ t > 0.$$

As examples, we can consider L_p, Orlicz, Lorentz and Marcinkiewicz spaces. All of them are permutation invariant Banach spaces of functions. Moreover, each permutation invariant Banach space E is symmetric with the Fatou norm $\|\cdot\|_E$ (i.e., $0 \le x_i \uparrow x \in E$ implies $\|x_i\|_E \uparrow \|x\|_E$). Let E be a quasi-Banach space on $[0, 1]$. Define a noncommutative symmetric space

$$L_E(\mathcal{M}) = \{x \in L_0(\mathcal{M}) : \mu.(x) \in E\}$$

with the norm $\|x\|_{L_E(\mathcal{M})} = \|\mu.(x)\|_E$. It is well-known that $(L_E(\mathcal{M}); \|\cdot\|_{L_E(\mathcal{M})})$ is a quasi-Banach space [9, 10]. Now let E be a quasi-Banach lattice and let $0 < r < \infty$. Then E is called r-convex (respectively, r-concave) if there exists a constant $C > 0$ such that for all finite sequences (x_n) in E the following inequalities are fulfilled:

$$\left\| \left(\sum_{k=1}^n |x_k|^r \right)^{1/r} \right\|_E \le C \left(\sum_{k=1}^n \|x_k\|_E^r \right)^{1/r}$$

$\left(\text{respectively, } \left(\sum\limits_{k=1}^n \|x_k\|_E^r \right)^{1/r} \le C \left\| \left(\sum_{k=1}^n |x_k|^r \right)^{1/r} \right\|_E \right)$. The best constant $C > 0$, as usually, is denoted as $M^{(r)}(E)$ (respectively, $M_{(r)}(E)$). Assume that $M^{r_1}(E) \le M^{r_2}(E)$ and $M_{r_2}(E) \le M_{r_1}(E)$ for $r_1 \le r_2$. For example, each space $L_p(\mu)$ is p-convex and p-concave with the constant 1, $M^{(2)}(L_p(\mu))=1$ for $2 \le p$ and $M_{(2)}(L_p(\mu)) = 1$ for $p \le 2$. For the notions of convexity and concavity see [11]. If $M^{\max(1,r)}(E) = 1$, then

$$E^r := \{x \in L_0(\Omega) : |x|^{1/r} \in E\},$$

equipped with the norm $\|x\|_{E^r} = \||x|^{1/r}\|_E^r$ is a Banach space of functions, i.e., $1/\min(1, r)$-convex. Therefore, for each operator $x \in L_0(\mathcal{M})$ it is true that $\mu(|x|^r) = \mu(x)^r$. Hence, for each symmetric Banach space of functions E on the interval $[0, 1]$ that satisfies the condition $M^{\max(1,r)}(E) = 1$ we can define [9]

$$L_{E^r}(\mathcal{M}) := \{x \in L_0(\mathcal{M}) : |x|^{1/r} \in L_E(\mathcal{M})\}$$

and $\|x\|_{L_{E^r}(\mathcal{M})} = \|\mu(|x|)\|_{E^r} = \|\mu(|x|^{1/r})\|_E^r = \||x|^{1/r}\|_{L_E(\mathcal{M})}^r$.

Definition 1. Let E be a symmetric quasi-Banach space on $[0, 1]$ and $\mathcal{A}$ be a finite subdiagonal subalgebra of von Neumann algebra $\mathcal{M}$. Then $H_E(\mathcal{A}) = \overline{\mathcal{A}}^{\|\cdot\|_{L_E(\mathcal{M})}}$ is called a symmetric Hardy space associated with $\mathcal{A}$. Denote $\overline{\mathcal{A}_0}^{\|\cdot\|_{L_E(\mathcal{M})}}$ by $H_E^0(\mathcal{A})$ (see [12]), where the bar means a closure.

The theory of vector-valued noncommutative space L_p was first considered by G. Pisier in [13] for the case when $\mathcal{M}$ is hyperfinite, later by M. Junge [14] (see also [15]) for the general case. Spaces $H_p^{(r,s)}(\mathcal{A};\ell_\infty)$ and $H_p(\mathcal{A};\ell_1)$ are considered in [16]. Noncommutative symmetric spaces $L_E(\mathcal{M};\ell_\infty)$ and $L_E(\mathcal{M};\ell_1)$ were introduced by A. Defant in [17] and S. Dirksen in [18]. Let us define analogs of these spaces in the same manner.

Definition 2. (i) Define a space $H_E(\mathcal{A},\ell_\infty)$ as the space of all sequences $x=(x_n)_{n\geq 1}$ in $H_E(\mathcal{A})$ that allows factorization of the following type: There exist $a,b\in H_{E^{1/2}}(\mathcal{A})$ and a bounded sequence $y=(y_n)\subset\mathcal{A}$ such that

$$x_n=ay_nb\ \ \forall n\geq 1.$$

Let $x\in H_E(\mathcal{A},\ell_\infty)$ be given, define

$$\|x\|_{H_E(\mathcal{A},\ell_\infty)}:=\inf\{\|a\|_{H_{E^{1/2}}(\mathcal{A})}\sup_n\|y_n\|_\infty\|b\|_{H_{E^{1/2}}(\mathcal{A})}\},$$

where the infimum is taken over all factorizations of (x_n). Moreover, define $H_E(\mathcal{A};\ell_\infty^c)$ (here c stands for "column") as the space of all $(x_n)_{n\geq1}$ in $H_E(\mathcal{A})$, for which there exist $b\in H_E(\mathcal{A})$ and a bounded sequence $(y_n)_{n\geq1}$ in $\mathcal{M}$ such that $x_n=y_nb$ and

$$\|x\|_{H_E(\mathcal{A},\ell_\infty)}:=\inf\{\sup_n\|y_n\|_\infty\|b\|_{H_E(\mathcal{A})}\}.$$

Also, define a row version $H_E(\mathcal{A};\ell_\infty^r)$ of all sequences that allows uniform factorization $x_n=ay_n$ with $a\in H_E(\mathcal{A})$ and $(y_n)_{n\geq1}$, that are uniformly bounded in $\mathcal{M}$.

(ii) Define $H_E(\mathcal{A};\ell_1)$ as the space of all sequences $x=(x_n)_{n\geq1}$ in $H_E(\mathcal{A})$ that can be decomposed as

$$x_n=\sum_{k=1}^\infty u_{kn}v_{nk}\ \ \forall n\geq1$$

for two families $(u_{kn})_{k,n\geq1}$ and $(v_{nk})_{n,k\geq1}$ from $H_{E^{1/2}}(\mathcal{A})$ such that

$$\sum_{k,n=1}^\infty u_{kn}u_{kn}^*\in L_E(\mathcal{M})\ \text{ and }\ \sum_{n,k=1}^\infty v_{nk}^*v_{nk}\in L_E(\mathcal{M}).$$

In this space, define the norm

$$\|x\|_{H_E(\mathcal{A};\ell_1)}:=\inf\left\{\left\|\sum_{k,n=1}^\infty u_{kn}u_{kn}^*\right\|_{H_E(\mathcal{A})}^{1/2}\cdot\left\|\sum_{n,k=1}^\infty v_{nk}^*v_{nk}\right\|_{H_E(\mathcal{A})}^{1/2}\right\},$$

where the infimum is taken over all factorizations of x.

2. PROPERTIES OF SPACES $H_E(\mathcal{A};\ell_\infty)$ AND $H_E(\mathcal{A};\ell_1)$

Lemma 1. (i) *Let E be a symmetric r-convex quasi-Banach space of functions on $[0,1]$ for some $0<r<\infty$ and E do not contain space c_0, where $c_0=\{(x_n)_{n\geq1}\subset\mathbb{C}:\lim\limits_{n\to\infty}x_n=0\}$. If $(x_n)\in L_E(\mathcal{M};\ell_\infty)$, then there exist $h,g\in H_{E^{1/2}}(\mathcal{A})$ and $(z_n)\subset\mathcal{M}$ such that $h^{-1},g^{-1}\in\mathcal{A}$, and for all n, $x_n=hz_ng$, and $\sup\limits_n\|z_n\|_\infty\leq1$. Moreover,*

$$\|(x_n)\|_{L_E(\mathcal{M};\ell_\infty)}=\inf\{\|h\|_{H_{E^{1/2}}(\mathcal{A})}\sup_n\|z_n\|_\infty\|g\|_{H_{E^{1/2}}(\mathcal{A})}\},\tag{1}$$

where the infimum is taken over all factorizations of $(x_n)_{n\geq1}$.

(ii) *Let E be a symmetric quasi-Banach space of functions on $[0,1]$, then $L_E(\mathcal{M};\ell_\infty)=L_E(\mathcal{M};\ell_\infty^r)\cdot L_E(\mathcal{M};\ell_\infty^c)$, where $L_E(\mathcal{M};\ell_\infty^r)\cdot L_E(\mathcal{M};\ell_\infty^c)=\{xy:x\in L_E(\mathcal{M};\ell_\infty^r),y\in L_E(\mathcal{M};\ell_\infty^c)\}$.*

Sketch of Proof. (i). If $x \in L_E(\mathcal{M};\ell_\infty)$, then for any $\varepsilon > 0$ there exists a bounded sequence $y = (y_n)$ in $\mathcal{M}$ and operators $a, b \in L_{E^{1/2}}(\mathcal{M})$, such that for each n

$$x_n = ay_nb, \quad \|y_n\| \le 1,$$

and $\|a\|_{L_{E^{1/2}}(\mathcal{M})}\|b\|_{L_{E^{1/2}}(\mathcal{M})} < \|x\|_{L_E(\mathcal{M};\ell_\infty)} + \varepsilon$. Let $a^* = u|a^*|$ and $b = v|b|$ be polar decompositions of operators a^* and b, respectively. Assume $c = (|a^*|^2 + \varepsilon)^{\frac{1}{2}}$ and $d = (|b|^2 + \varepsilon)^{\frac{1}{2}}$. It is clear that $|a^*|^2 \le c^2$ and $|b|^2 \le d^2$. Then by remark 2.3 from [19] there exist the shrinking operators $\omega, \theta \in \mathcal{M}$ such that $|a^*| = \omega c$, $|b| = \theta d$. Since $c, d \in L_{E^{1/2}}(\mathcal{M})$ and $c^{-1}, d^{-1} \in \mathcal{M}$, by proposition 4.3 (i) from [12] there exist $h, g \in H_{E^{1/2}}(\mathcal{A})$ and unitary operators $\nu, w \in \mathcal{M}$ such that $c = h\nu$, $d = wg$, and $h^{-1}, g^{-1} \in \mathcal{A}$. It is obvious that $x_n = h[\nu\omega^* u^* y_n v\theta w]g$. Suppose $z_n = \nu\omega^* u^* y_n v\theta w$, then it is clear that $\sup\limits_n \|z_n\|_\infty \le 1$. Equality (1) is now obvious.

(ii). Let $x \in L_E(\mathcal{M};\ell_\infty)$, then $x_n = ay_nb$. By picking $x_n^{(1)} = a$ and $x_n^{(2)} = y_nb$ for all n, we obtain

$$x_n = x_n^{(1)}x_n^{(2)} \ \ \forall n$$

and $(x_n^{(1)}) \in L_E(\mathcal{M};\ell_\infty^r)$, $(x_n^{(2)}) \in L_E(\mathcal{M};\ell_\infty^c)$. □

Proposition 1. *Let E be a symmetric r-convex quasi-Banach space of functions on $[0,1]$ for some $0 < r < \infty$ and E do not contain space c_0. Then*

$$H_E(\mathcal{A};\ell_\infty) = \{(x_n) \in L_E(\mathcal{M};\ell_\infty) : (x_n) \subset H_E(\mathcal{A})\}.$$

Moreover,

$$\|(x_n)\|_{L_E(\mathcal{M};\ell_\infty)} = \|(x_n)\|_{H_E(\mathcal{A};\ell_\infty)} \ \ \forall (x_n) \in H_E(\mathcal{A};\ell_\infty).$$

Theorem 1. *Let E be a symmetric r-convex quasi-Banach space of functions on $[0,1]$ for some $0 < r < \infty$ and E do not contain space c_0. Then the space $H_E(\mathcal{A},\ell_\infty)$ is quasi-Banach one.*

Corollary 1. Let the conditions of Theorem 1 be fulfilled. Then

$$H_E(\mathcal{A};\ell_\infty) = H_E(\mathcal{A};\ell_\infty^r) \cdot H_E(\mathcal{A};\ell_\infty^c).$$

Lemma 2. *Let the conditions of Theorem 1 be fulfilled. If $x \in L_E(\mathcal{M};\ell_1)$, then for any n there exist $(a_{kn})_{k\ge1} \subset H_{E^{1/2}}(\mathcal{A})$, $(b_{nk})_{k\ge1} \subset H_{E^{1/2}}(\mathcal{A})$ and $(y_{nk})_{k\ge1} \subset \mathcal{M}$, such that*

$$x_n = \sum_{k=1}^{\infty} a_{kn}y_{nk}b_{nk},$$

where $(a_{kn}^{-1})_{k\ge1}, (b_{nk}^{-1})_{k\ge1} \subset \mathcal{A}$, and $\sup_n \|y_{nk}\|_\infty \le 1$ for any n and k. Moreover,

$$\|x\|_{L_E(\mathcal{M};\ell_1)} = \inf\left\{\left\|\sum_{k,n=1}^{\infty} a_{kn}a_{kn}^*\right\|_{H_E(\mathcal{A})}^{1/2} \cdot \sup_n \|y_{nk}\|_\infty \cdot \left\|\sum_{n,k=1}^{\infty} b_{nk}^*b_{nk}\right\|_{H_E(\mathcal{A})}^{1/2}\right\},$$

where the infimum is taken over all factorizations of x.

Similarly to Proposition 1, we obtain

Proposition 2. *Let the conditions of Theorem 1 be fulfilled. Then*

$$H_E(\mathcal{A};\ell_1) = \{(x_n) \in L_E(\mathcal{M};\ell_1) : (x_n) \subset H_E(\mathcal{A})\}.$$

Moreover,

$$\|(x_n)\|_{L_E(\mathcal{M};\ell_1)} = \|(x_n)\|_{H_E(\mathcal{A};\ell_1)} \ \ \forall (x_n) \in H_E(\mathcal{A};\ell_1).$$

Lemma 2 and Proposition 2 yield

Theorem 2. *Let the conditions of Theorem* 1 *be fulfilled. Then the space* $H_p(\mathcal{A};\ell_1)$ *is quasi-Banach one.*

Proposition 3. *Let the conditions of Theorem* 1 *be fulfilled. Then*

$$H_E(\mathcal{A};\ell_\infty)=H_r(\mathcal{A};\ell_\infty)\cap L_E(\mathcal{M};\ell_\infty),\ \ H_E^0(\mathcal{A};\ell_\infty)=H_r^0(\mathcal{A};\ell_\infty)\cap L_E(\mathcal{M};\ell_\infty).$$

Proposition 4. *Let the conditions of Theorem* 1 *be fulfilled. Then*

$$H_E(\mathcal{A};\ell_1)=H_r(\mathcal{A};\ell_1)\cap L_E(\mathcal{M};\ell_1),\ \ H_E^0(\mathcal{A};\ell_1)=H_r^0(\mathcal{A};\ell_1)\cap L_E(\mathcal{M};\ell_1).$$

Proposition 5. *Let the conditions of Theorem* 1 *be fulfilled. Then*

$$H_E(\mathcal{A};\ell_\infty)=\{x\in L_E(\mathcal{M};\ell_\infty):\tau(x_nc)=0\ \ \forall c\in\mathcal{A}_0,\ \ \forall n\}.$$

Moreover,

$$H_E^0(\mathcal{A};\ell_\infty)=\{x\in L_E(\mathcal{M};\ell_\infty):\tau(x_nc)=0\ \ \forall c\in\mathcal{A},\ \ \forall n\},$$

here $H_E^0(\mathcal{A};\ell_\infty)=\{x\in H_E(\mathcal{A};\ell_\infty):\Phi(x_n)=0\ \ \forall n\}$.

3. SHRINKINGS Φ ON THE SPACES $H_E(\mathcal{A};\ell_\infty)$ AND $H_E(\mathcal{A};\ell_1)$

It is a well-known fact that Φ can be extended to a shrinking projector from $L_p(\mathcal{M})$ to $L_p(\mathcal{D})$ for all $1\le p\le\infty$. In general case, Φ cannot be continuously extended to $L_p(\mathcal{M})$ for $p<1$. However, in [20] it is proved that Φ is a shrinking projector from $H_p(\mathcal{A})$ to $L_p(\mathcal{D})$ for the case of $p<1$, and in [12] Φ is extended to a shrinking projector on $H_E(\mathcal{A})$.

Theorem 3. *Let* E *be a symmetric quasi-Banach space of functions on* $[0,1]$ *with* $M^{(r)}(E)=1$ *for some* $0<r<\infty$, *and let* $(h_n)\subset H_E(\mathcal{A})$. *Define* $(h_n)\mapsto(\Phi(h_n))$, *then* Φ *can be extended to a shrinking projector from* $H_E(\mathcal{A};\ell_\infty)$ *to* $L_E(\mathcal{D};\ell_\infty)$, *i.e.,*

$$\|\Phi(h)\|_{L_E(\mathcal{D};\ell_\infty)}\le\|h\|_{H_E(\mathcal{A};\ell_\infty)}$$

for all $h\in H_E(\mathcal{A};\ell_\infty)$. *The extension is also denoted as* Φ.

Sketch of Proof. Let $(h_n)\in H_E(\mathcal{A};\ell_\infty)$, then for any $\varepsilon>0$ there exist $a,b\in H_{E^{1/2}}(\mathcal{A})$ and a bounded sequence $(x_n)\subset\mathcal{A}$ such that for all n, $h_n=ax_nb$, and

$$\|(h_n)\|_{H_E(\mathcal{A};\ell_\infty)}+\varepsilon\ge\|a\|_{H_{E^{1/2}}(\mathcal{A})}\sup_n\|x_n\|_\infty\|b\|_{H_{E^{1/2}}(\mathcal{A})}.$$

Then by corollary 3.3 and theorem 3.2 from [12] we have

$$\Phi(h_n)=\Phi(ax_nb)=\Phi(a)\Phi(x_n)\Phi(b),$$

where $\Phi(a)\in L_E(\mathcal{D})$, $\Phi(x_n)\in\mathcal{D}$, $\Phi(b)\in L_E(D)$ and

$$\|\Phi(a)\|_{L_{E^{1/2}}(\mathcal{D})}\le\|a\|_{H_{E^{1/2}}(\mathcal{A})},\ \ \|\Phi(x_n)\|_\infty\le\|x_n\|_\infty,\ \ \|\Phi(b)\|_{L_{E^{1/2}}(\mathcal{D})}\le\|b\|_{H_{E^{1/2}}(\mathcal{A})}.$$

Thus, we obtain

$$\begin{aligned}\|(\Phi(h_n))\|_{L_E(\mathcal{D};\ell_\infty)}&\le\|\Phi(a)\|_{L_{E^{1/2}}(\mathcal{D})}\sup_n\|\Phi(x_n)\|_{L_\infty(\mathcal{D})}\|\Phi(b)\|_{L_{E^{1/2}}(\mathcal{D})}\\&\le\|a\|_{H_{E^{1/2}}(\mathcal{A})}\sup_n\|x_n\|_{H_\infty(\mathcal{A})}\|b\|_{H_{E^{1/2}}(\mathcal{A})}\le\|(h_n)\|_{H_E(\mathcal{A};\ell_\infty)}+\varepsilon.\end{aligned}$$

Passing to the limit as $\varepsilon\to0$, we obtain the necessary inequality. □

Theorem 4. *Let* E *be a symmetric quasi-Banach space of functions on* $[0,1]$ *with* $M^{(r)}(E)=1$ *for all* $0<r<\infty$, *and let* $(y_n)\subset H_E(\mathcal{A})$. *Define* $(y_n)\mapsto(\Phi(y_n))$, *then* Φ *can be extended to a shrinking projector from* $H_E(\mathcal{A};\ell_1)$ *to* $L_E(\mathcal{D};\ell_1)$, *i.e.,*

$$\|\Phi(y)\|_{L_E(\mathcal{D};\ell_1)}\le\|y\|_{H_E(\mathcal{A};\ell_1)}$$

for all $y\in H_E(\mathcal{A};\ell_1)$. *The extension is also denoted as* Φ.

ACKNOWLEDGMENTS

The author wishes to express his deep gratitude to A. M. Bikchentaev for his valuable comments.

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Translated by P. A. Novikov