ISSN 1066-369X, Russian Mathematics (Iz. VUZ), 2015, Vol. 59, No. 11, pp. 74–79. © Allerton Press, Inc., 2015.
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Noncommutative Vector-Valued Symmetric Hardy Spaces

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Received March 10, 2015

Abstract—We introduce noncommutative vector-valued symmetric Hardy spaces and demonstrate that they are quasi-Banach ones. For these spaces we prove analogs of some basic properties of the Hardy space.

DOI: 10.3103/S1066369X15110092

Keywords: *von Neumann algebra, subdiagonal algebra, noncommutative vector-valued symmetric Hardy space, conditional expectation.*

1. INTRODUCTION

Let $\mathcal{M}$, a finite von Neumann algebra with an exact normal trace τ act in the Hilbert space $\mathcal{H}$. Let $\mathcal{D}$ be a subalgebra of von Neumann algebra $\mathcal{M}$, $\Phi : \mathcal{M} \to \mathcal{D}$ be a unique normal exact conditional expectation such that $\tau \circ \Phi = \tau$. Then an $*$-weakly closed subalgebra $\mathcal{A}$ of the algebra $\mathcal{M}$ with respect to Φ is a finite subdiagonal algebra [1], if the following conditions are fulfilled:

(i) $\mathcal{A} + \mathcal{A}^*$ is $*$-weakly closed in $\mathcal{M}$;

(ii) Φ is multiplicative on $\mathcal{A}$, i.e., $\Phi(ab) = \Phi(a)\Phi(b) \ \ \forall a, b \in \mathcal{A}$;

(iii) $\mathcal{A} \cap \mathcal{A}^* = \mathcal{D}$, where $\mathcal{A}^* = \{a^* : a \in \mathcal{A}\}$.

An algebra $\mathcal{D}$ is said to be a diagonal algebra for $\mathcal{A}$. In [2] it is proved that each finite subdiagonal algebra $\mathcal{A}$ is maximal. This means that if there exists another algebra $\mathcal{B}$ that contains $\mathcal{A}$ with respect to Φ, then $\mathcal{B} = \mathcal{A}$. Let $0 < p \leq \infty$. By $L_p(\mathcal{M})$ let us denote a noncommutative L_p-space associated with $(\mathcal{M}, \tau)$. For $L_\infty(\mathcal{M})$, let us take $\mathcal{M}$ with an operator norm [3]. By $\|\cdot\|_p$ denote the norm of the space $L_p(\mathcal{M})$. For $p < \infty$ we define $H_p(\mathcal{A})$ as a closure of $\mathcal{A}$ with respect to the norm $L_p(\mathcal{M})$, and assume $H_\infty(\mathcal{A}) = \mathcal{A}$ for $p = \infty$. These spaces are also called Hardy spaces associated with $\mathcal{A}$. They are noncommutative analogs of classical Hardy spaces on a torus $\mathbb{T}$. Hardy spaces are in detail covered in surveys [1], [3] and [4]. Some properties of the space $H_p(\mathcal{A})$ were considered in the work of G. Sh. Skvortsova [5]. After the pioneering work by W. B. Arveson [4] a great interest to noncommutative Hardy spaces emerged. For more detail, see [3, 6, 7].

A closed everywhere densely defined in $\mathcal{H}$ operator x with a domain $D(x)$ is called associated with $\mathcal{M}$ ($x\eta\mathcal{M}$), if and only if $ux = xu$ for all unitary u that belong to the commutant $\mathcal{M}'$ of the algebra $\mathcal{M}$. An operator $x\eta\mathcal{M}$ is called τ-measurable if for an arbitrary $\varepsilon > 0$ there exists a projector $e \in \mathcal{M}$ such that $e(\mathcal{H}) \subseteq D(x)$ and $\tau(1 - e) < \varepsilon$. Denote the set of all τ-measurable operators as $L_0(\mathcal{M})$. A set $L_0(\mathcal{M})$ is an $*$-algebra with strong sum and product, that are closures of the algebraic sum and multiplication [3]. It is a well-known fact that for each operator $x\eta\mathcal{M}$ in $\mathcal{H}$, all spectral projectors $e_s^\perp(|x|) = \chi_{(s,\infty)}(|x|)$, corresponding to the interval (s, ∞), belong to $\mathcal{M}$, and $x \in L_0(\mathcal{M})$ if and only if $\tau(e_s^\perp(|x|)) < \infty$ for some $s \in \mathbb{R}$. Define a non-increasing permutation (or generalized singular values) of the operator $x \in L_0(\mathcal{M})$ in the following way:

$$\mu_t(x) = \inf\{s > 0 : \lambda_s(x) \leq t\}, \quad t > 0,$$

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