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Derivations of a Matrix Ring Containing a Subring of Triangular Matrices

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Abstract—We describe the derivations (including Jordan and Lie ones) of finitary matrix rings, containing a subring of triangular matrices, over an arbitrary associative ring with identity element.

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INTRODUCTION

An endomorphism φ of an additive group of a ring K is called a derivation if $\varphi(ab) = \varphi(a)b + a\varphi(b)$ for any $a, b \in K$. Derivations of the Lie and Jordan rings associated with K are called, respectively, the Lie and Jordan derivations of the ring K; they are denoted, respectively, by $\Lambda(K)$ and $J(K)$. (We obtain them by replacing the multiplication in K with the Lie multiplication $a * b = ab - ba$ and the Jordan one $a \circ b = ab + ba$, respectively.) In 1957 I. N. Herstein proved [1] that every Jordan derivation of a primary ring of characteristic $\neq 2$ is a derivation. His theorem was generalized by several authors ([2–9] et al.) for other classes of rings and algebras.

Let R be an arbitrary associative unital ring and let Γ be a linearly ordered set (a chain) with the order relation $\leqslant$. Following [10] (see also [11]), we understand a carpet as the set $I = \{I_{ij} \mid i, j \in \Gamma\}$ of ideals of the ring R such that $I_{ij} I_{jk} \subseteq I_{ik}$ for any $i, j, k \in \Gamma$. We denote by S_I the ring of finitary (i.e., with a finite number of nonzero elements) Γ-matrices $\|a_{ij}\|$, $a_{ij} \in I_{ij}$, $i, j \in \Gamma$. Putting $I_{ij} = R$ for all $i > j$ and $I_{ij} = 0$ for $i \leqslant j$, we get the ring of nil-triangular finitary Γ-matrices; its Lie, Jordan, and usual derivations are studied in [7].

Let $I_{ij} = R$ for all $i \geqslant j$. Then S_I is an intermediate ring between the ring of all finitary Γ-matrices and its subring of triangular finitary Γ-matrices. As is proved in [5], for a given finite chain Γ an arbitrary derivation of S_I is the sum of an inner derivation (i.e., that in the form $x \to xa - ax$, where $x \in S_I$ and a is a fixed matrix in S_I) and an induced one (i.e., that in the form $\|a_{ij}\| \to \|\varphi(a_{ij})\|$, where φ is a derivation of the ring R). In [5] one also describes all Jordan derivations of the ring S_I in the case $2R = R$. Note that in the case of an infinite chain Γ the mapping $x \to xa - ax$, $x \in S_I$, is a derivation of the ring S_I, even if the matrix $a = \|a_{ij}\|$, $a_{ij} \in I_{ij}$, is weakly finitary, i.e., each row and each column of the matrix a contains a finite number of nonzero elements. We call such derivations locally inner ones. In the following three theorems we describe the usual, Jordan, and Lie derivations of the ring S_I without any assumptions about the invertibility of the number 2 and the finiteness of the chain Γ.

Theorem 1. *Any derivation of the ring S_I for $|\Gamma| \geqslant 2$ is the sum of locally inner and induced derivations.*

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