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Properties of the Coxeter Transformations for the Affine Dynkin Cycle

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Abstract—We study spectral properties of the Coxeter transformations for the affine Dynkin cycle and find the Jordan form of the Coxeter transformation and the Coxeter numbers.

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Introduction. Let W be the Weyl group generated by the linear transformations in the vector space E_Γ associated to a graph $\Gamma = (\Gamma_0, \Gamma_1)$ (Γ_0 is the set of vertices, Γ_1 is the set of edges) so that each coordinate of a vector corresponds to a vertex of the graph and the set of generators S of the group W consists of reflections s_α associated to vertices α of Γ which act by the rule

$$\forall x \in E_\Gamma \ (s_\alpha x)_\beta = x_\beta \ \text{for} \ \alpha \neq \beta; \quad (s_\beta x)_\beta = -x_\beta + \sum_{\{\alpha,\beta\}\in\Gamma_1} x_\alpha.$$

Let B_Γ denote the quadratic form on the space E_Γ defined by

$$B_\Gamma(x) = \sum_{\alpha\in\Gamma_0} (x_\alpha)^2 - \sum_{\{\alpha,\beta\}\in\Gamma_1} x_\alpha x_\beta.$$

The Coxeter transformations (CT) $c = s_n \dots s_2 s_1$ defined for bijections

$$S \to \{1, 2, \dots, n\} \tag{1}$$

play an important part in the study and use of the Weyl group. The properties of CT's are well-studied [1–4] for all connected Coxeter graphs with $B_\Gamma(x) > 0$ (Dynkin graphs) and with $B_\Gamma(x) \geq 0$ (affine (completed) Dynkin graphs) except for the affine cycle $\widetilde{A}_n$ (simple cycle). In [1] (P. 146) it is proved that all CT's for graphs without cycles, independently of the choice of bijection (1), are conjugate in the group W. Therefore, their spectral properties coincide. For the graph $\widetilde{A}_n$, it is not the case, which gives no possibility to transfer completely the results from [1–4].

In this paper, we solve the problem of spectral properties of CT's for the cycle $\widetilde{A}_n$, $n \geq 1$.

1^0. For each graph Γ, bijection (1) defines its orientation Λ such that the arrow of an edge is directed from the greater number to the smaller one. From Lemma 1.2 in [2] it follows that all CT's corresponding to the same orientation coincide. For this reason, we denote a CT by c_Λ.

Let us introduce a direction of going around the cycle and denote by $k(\Lambda)$ the number of arrows which agree with the chosen direction. In what follows we consider only cycles for which $0 < k(\Lambda) < n + 1$.

Lemma 1. *If $k(\Lambda) = k(\Lambda')$, then c_Λ and $c_{\Lambda'}$ are conjugate in W.*

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