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Correctness of Cauchy-Type Problems for Abstract Differential Equations with Fractional Derivatives

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Abstract—We prove the uniform correctness of a Cauchy-type problem with two fractional derivatives and a bounded operator A. We propose a criterion for the uniform correctness of unbounded operator A.

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Let $M_k^{\alpha,\beta}$ be a differential operator in the form $M_k^{\alpha,\beta} = D^\alpha\left(t^k D^\beta\right)$ with left-hand fractional Riemann–Liouville derivatives of orders $\alpha \in (0,1)$ and $\beta \in (0,1)$ ([1], P. 84)

$$D^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-s)^{-\alpha}u(s)ds,\ \ t \in (0,\infty);$$

here $\Gamma(\cdot)$ is the Euler gamma function.

In a Banach space X we consider the following Cauchy-type problem with a linear closed operator A:

$$M_k^{\alpha,\beta}u(t) = t^\gamma Au(t),\ \ t > 0, \tag{1}$$

$$\lim_{t\to 0} D^{\beta-1}u(t) = u_0,\ \ \lim_{t\to 0} D^{\alpha-1}\left(t^k D^\beta u(t)\right) = u_1, \tag{2}$$

where

$$D^{\beta-1}u(t) = I^{1-\beta}u(t) = \frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}u(s)ds$$

is the left-hand fractional Riemann–Liouville integral of order $1-\beta$.

The definition of initial conditions (2) has the following specificity: In the first condition the total order of derivatives is less by 1 than that in Eq. (1), in the second condition the difference of orders increases by α.

Moreover, the peculiarity of the considered problem consists in the presence of two conditions in the form (2) even if $0 < \alpha+\beta < 1$. In the case of an ordinary differential equation of fractional order ($E = \mathbf{R}$, A is the operator of multiplication by a number) this peculiarity can be explained by the equality ([1], formula (2.68))

$$D^\alpha D^\beta u\,(t) = D^{\alpha+\beta}u\,(t) - \frac{D^{\beta-1}u\,(0)\,t^{-\alpha-1}}{\Gamma(-\alpha)}.$$

This equality allows one to reduce Eq. (1) with $k=\gamma=0$ to the following inhomogeneous equation:

$$D^{\alpha+\beta}u\,(t) = Au(t) + \frac{D^{\beta-1}u\,(0)\,t^{-\alpha-1}}{\Gamma(-\alpha)}.$$

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