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SOLVABILITY OF THE TWO-POINT BOUNDARY-VALUE PROBLEM FOR A LINEAR SINGULAR FUNCTIONAL DIFFERENTIAL EQUATION

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1. Introduction

In this paper, we obtain conditions under which a solution x of the problem

$$(t-a)^{\mu_1}(b-t)^{\mu_2}\ddot{x}(t) = (Tx)(t) + f(t), \quad t \in [a,b],$$
$$x(a) = 0, \quad x(b) = 0,$$

exists, is unique and satisfies the inequality

$$x(t) < 0 \text{ for all } t \in (a,b) \tag{1}$$

for any nonnegative almost everywhere (a. e.) $f \in L[a,b]$, $f \not\equiv 0$.

In [1], the necessary conditions for inequality (1) were formulated. Here we supplement the results of [1] with the general sufficient condition and the effective coefficient characteristics. We use the terminology and the notation from [1].

Let us consider the boundary-value problem

$$\begin{gathered}(\mathcal{L}x)(t) \stackrel{\text{def}}{=} \pi(t)\ddot{x}(t) - (Tx)(t) = f(t), \quad t \in [a,b],\\ x(a) = \alpha, \quad x(b) = \beta,\end{gathered} \tag{2}$$

where $\pi(t) = (t-a)^{\mu_1}(b-t)^{\mu_2}$, $\mu_1, \mu_2 \in [0,1]$. We assume that f belongs to $L[a,b]$, which is a Banach space of real functions summable on $[a,b]$; T is a linear bounded operator acting from the Banach space of real continuous functions $C[a,b]$ to $L[a,b]$. The equation $\mathcal{L}x = f$ is called singular [2], [3] if at least one number μ_1 or μ_2 is not equal to zero.

We will seek a solution of problem (2) in the Banach space D defined in [1]. Note that in the space D the problem

$$(\mathcal{L}_0 x)(t) \stackrel{\text{def}}{=} \pi(t)\ddot{x}(t) = f(t), \quad t \in [a,b],$$
$$x(a) = \alpha, \quad x(b) = \beta,$$

has a unique solution

$$x(t) = (G_0 f)(t) + \alpha\frac{t-a}{b-a} + \beta\frac{b-t}{b-a}, \quad t \in [a,b],$$

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