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Generatrices and Relations of a Generalized Orthogonal Group Over Commutative Semilocal Rings without Identity

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1. INTRODUCTION

The description of orthogonal groups (and close to them ones) in terms of generatrices and relations represents one of the main problems of the combinatorial theory of linear groups. This paper is dedicated to the mentioned question; namely, here we find generating elements and defining relations of a generalized orthogonal group $O^\circ(n, R)$, $n \geq 2$, over a commutative semilocal ring R (in general, without identity) under certain natural constraints.

In order to formulate the problem exactly, let us define several necessary notions. Let Λ be an arbitrary associative ring and let $\circ$ be its quasi-multiplication (i. e., $x \circ y = x + xy + y$). An element α from Λ is said to be *quasi-invertible*, if with certain $\alpha' \in \Lambda$, $\alpha \circ \alpha' = 0 = \alpha' \circ \alpha$. If α is quasi-invertible, then one can uniquely determine its quasi-inverse α'. The set of all quasi-invertible elements Λ° from Λ forms a group with respect to the composition $\circ$ (where the identity is zero). We consider the case when $\Lambda = M(n, R)$ is a complete matrix ring over a ring T which is (associative-)commutative and not necessarily with identity. Let $^\top$ mean the transposition in Λ. The set of quasi-invertible matrices from Λ such that $a' = a^\top$ forms a group with respect to the matrix quasi-multiplication. We denote it by $O^\circ(n, R)$ and call it a generalized orthogonal group of the degree n over the ring R. Note that if R contains the identity, then the mapping $O(n, R) \to O^\circ(n, R)$, $E + a \to a$, where E is the unit matrix of the order n, defines an isomorphism. So the introduced group $O^\circ(n, R)$ generalizes the notion of a usual orthogonal group to the most general cases of associative-commutative rings R.

Let now R be a commutative semilocal ring (not necessarily with identity) and let $J = J(R)$ be its Jacobson radical. By definition it means that $R/J \cong k_1 \oplus \cdots \oplus k_m$, where k_i are certain fields ($i = 1, \ldots, m$). Let R_i stand for the complete preimage of the addend k_i under the natural epimorphism

$$R \to \overline{R} = R/J, \quad x \to \overline{x} = x + J. \tag{1}$$

One can easily see that the ring R admits the decomposition

$$R = R_1 + \cdots + R_m. \tag{2}$$

The addends R_i in this decomposition form (commutative and not necessarily with identity) local subrings in R. For a subring Λ of the ring R (including the case $\Lambda = R$) we set $\Lambda^{(2)} = \{\alpha \circ \alpha : \alpha \in \Lambda^\circ\}$. For a nonempty subset $X \subseteq R$ we also use the notation $X^2 = \{x^2 : x \in X\}$. Further, we call a pair $\langle x, y \rangle$ from $R \times R$ *consistent*, if $x \circ x + y^2 = 0$. Let e_i stand for a certain (arbitrary) preimage of identity $1_i \in k_i$ under epimorphism (1) ($i = 1, \ldots, m$). The objective of this paper is to find generatrices and to define the relations of the generalized orthogonal group $O^\circ(n, R)$, $n \geq 2$, over, in general, a commutative semilocal ring R without identity, provided that

$$R_i^{(2)} + R_i^2 \subseteq R_i^{(2)}; \tag{R_i}$$

$$\text{there exist consistent pairs } \langle -e_i, y \rangle \in R \times R, \ i = 1, \ldots, m; \tag{$-e_i$}$$

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