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Solvability of the Goursat Problem in Quadratures

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Abstract—We find new variants of Goursat problem solution in quadratures on the basis of the combination of the Riemann method and the cascade integration. The results are applied to two Volterra equations with particular integrals.

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We consider the problem on evaluation of a regular solution to the equation

$$u_{xy} + au_x + bu_y + cu = f \tag{1}$$

in the domain $D = \{x_0 < x < x_1,\ y_0 < y < y_1\}$ subject to $u(x_0, y) = \varphi(y)$, $u(x, y_0) = \psi(x)$, $\varphi(y_0) = \psi(x_0)$, $x \in [x_0, x_1]$, $y \in [y_0, y_1]$.

As is known ([1], P. 172; [2], P. 147), we can solve this problem by the Riemann method, namely,

$$\begin{aligned} u(x,y) = R(x,y_0,x,y)\psi(x) + R(x_0,y,x,y)\varphi(y) - R(x_0,y_0,x,y)\psi(x_0) \\ + \int_{x_0}^{x}\left[b(\alpha,y_0)R(\alpha,y_0,x,y) - \frac{\partial}{\partial\alpha}R(\alpha,y_0,x,y)\right]\psi(\alpha)d\alpha \\ + \int_{y_0}^{y}\left[a(x_0,\beta)R(x_0,\beta,x,y) - \frac{\partial}{\partial\beta}R(x_0,\beta,x,y)\right]\varphi(\beta)d\beta + \int_{x_0}^{x}\int_{y_0}^{y}R(\alpha,\beta,x,y)f(\alpha,\beta)d\beta\, d\alpha. \end{aligned} \tag{2}$$

The problem is solvable in quadratures if the corresponding Riemann function R can be written explicitly. We know only few such cases ([2], pp. 15–20; [3]). The goal of this paper consists in obtaining new variants of the mentioned type. We apply the obtained results for solving the Volterra integral equations.

1. Our consideration is based on the Laplace method of cascade integration. As is known ([4], pp. 177–181), the constructions $h = a_x + ab - c$, $k = b_y + ab - c$, play an essential role for this method. If $k \equiv 0$ or $h \equiv 0$, then the Riemann function $R(x, y; \xi, \eta)$ for Eq. (1) can be obtained explicitly ([2], formulas (1.23) and (1.24)). But if $h \neq 0$ or $k \neq 0$, then we can write an equation in the form (1), namely,

$$\frac{\partial^2 u_{\pm1}}{\partial x \partial y} + a_{\pm1}\frac{\partial u_{\pm1}}{\partial x} + b_{\pm1}\frac{\partial u_{\pm1}}{\partial y} + c_{\pm1}u_{\pm1} = f_{\pm1}, \tag{3}$$

where coefficients $a_{\pm1}$, $b_{\pm1}$, and $c_{\pm1}$ are determined by formulas (3) from [4] (P. 179), and

$$f_1 = [a - (\ln h)_y]f - f_y, \quad f_{-1} = [b - (\ln k)_x]f - f_x. \tag{4}$$

In (4) and below the subscript +1 is written as “1”. The values h and k in (3) are replaced with

$$h_1 = 2h - k - (\ln h)_{xy}, \quad k_1 = h; \quad h_{-1} = k, \quad k_{-1} = 2k - h - (\ln k)_{xy}.$$

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