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Matrix Bernoulli Equations. II

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Abstract—In this paper, we find sufficient conditions for the solvability by quadratures of J. Bernoulli's equation defined over the set M_2 of square matrices of order 2. We consider the cases when such equations are stated in terms of bases of a two-dimensional abelian algebra and a three-dimensional solvable Lie algebra over M_2. We adduce an example of the third degree J. Bernoulli's equation over a commutative algebra.

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In this paper, a continuation of [1], we establish sufficient conditions for solvability by quadratures of J. Bernoulli's equation defined over the set of 2×2-matrices.

1. Let M_2 be the set of square 2×2-matrices with unity E over the field R of real numbers, $M_2^k(t)$ the set of 2×2-matrices whose components are functions of class $C^k(t)$ of a variable t ($t \in R$). Let L_N be a representation in M_2 of an N-dimensional Lie algebra ($N \leq 4$), $L_N^k(t) = L_N \cap M_2^k(t)$, and let E_α and $C_{\alpha\beta}^\gamma$ be, respectively, the basis matrices and the structure constants of L_N ($\alpha, \beta, \gamma = \overline{1, N}$). Then a generalization of the well-known in the theory of scalar ODEs J. Bernoulli's equation

$$x' \equiv \frac{d}{dt}x = a(t)x + b(t)x^m, \ \ \big(a(t), b(t)\big) \subset C^0(t), \ \ b(t) \not\equiv 0, \ \ m \neq 0, 1, \tag{1}$$

can be given in the form

$$X' = A_1X + XA_2 + XB_1X \cdots XB_{m-1}X, \ \ (A_k, B_l) \subset M_2^0(t), \ \ 0 \not\equiv X \in M_2^1(t), \tag{2}$$

where m is a natural number different from 0 and 1.

In [1] they proved theorem 3 which states that if the parameters of a matrix Bernoulli equation (MBE) (2) A_k ($k = 1, 2$) and B_l ($l = \overline{1, m-1}$) are continuous and belong to a representation over the set M_n of $n \times n$-matrices of a solvable Lie algebra L_N ($N \leq n^2$), then it is solvable by quadratures, and, for unsolvable Lie algebras L_N, this is possible only when A_k and B_l are of a special form.

In the case of M_2, this statement allows ones to establish the most general form of MBEs admitting exact integration. The consideration of this case is of interest, first, because of the possibilities to use it in modeling of real processes in the plane, second, because it presents a sufficiently complete demonstration of the main results of [1], and, third, because of the possibility to study the processes over M_n described by Bernoulli equations whose matrix parameters are given in terms of a basis of L_N isomorphic to $L_N \cap M_2$.

2. As is known, the full matrix algebra M_2 is four-dimensional and has a simple three-dimensional subalgebra $sl(2)$, which implies, according to the Lie–Engel criterion, that it is unsolvable. The basis of $sl(2)$ ([2], P. 284) consists of the matrices

$$E_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3}$$

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