Russian Mathematics (Iz. VUZ) *Izvestiya VUZ. Matematika*
Vol. 46, No. 11, pp.24–28, 2002 *UDC 514.76*

THE HAT THEOREM AND PROBLEMS OF CLASSIFICATION OF STRUCTURES ON RIEMANNIAN MANIFOLDS

A.A. Yermolitskiĭ

In this article we consider the hat theorem which allows to deform a subbundle in a principal G-bundle over a differentiable manifold. Further, using the notion of the second fundamental tensor field of a G-structure on a Riemannian manifold, introduced earlier, we study the problem of classification of these structures. We prove a theorem on "algebraization" of the classification problem and state an example of an almost Hermitian structure which belongs to different classes in different points of the manifold.

1. The hat theorem and a deformation of a subbundle in a principal G-bundle

1°. Let M be a connected manifold, $P(M,G,\pi)$ be a principal fiber bundle over M. Let us introduce a Riemannian metric g on M and consider the geodesic balls $B(p;R/2)\subset B(p;R)\subset U$ centered at $p\in M$, where U is a local chart of $P(M,G,\pi)$. Thus a diffeomorphism $\psi:\pi^{-1}(U)\to U\times G: u\mapsto(\pi(u),\varphi(u))$ exists, where $\varphi:\pi^{-1}(U)\mapsto G$, $\varphi(ua)=\varphi(u)a$ for all $u\in\pi^{-1}(U)$ and $a\in G$.

If $\psi^{-1}:U\times G\to\pi^{-1}(U)$ is the inverse mapping, then $\psi^{-1}|U\times\{e\}$ determines a section $s_1:U\to\pi^{-1}(U)$. Consider a section $s_2:U\to\pi^{-1}(U)$ such that $s_1(p)=s_2(p)$. Let $v:U\to G$, $x\mapsto v(x)$, be a mapping defined by the relation $\varphi(s_2(x))=\varphi(s_1(x)v(x))$ and $\exp_p$ be the exponential mapping of the Riemannian connection ∇ at p. Then the mapping $\exp_p^{-1}$ is well-defined on $\overline{B}(p;R)$ and one can consider the following section s over U:

$$s(x)=\begin{cases} s_1(x), & x\in U\setminus\overline{B}(p;R);\\ s_2(x)\left(v\left[\exp_p\left(\frac{2t-R}{t}\exp_p^{-1}(x)\right)\right]\right)^{-1}, & x\in\overline{B}(p;R)\setminus\overline{B}(p;R/2);\\ s_2(x), & x\in\overline{B}(p;R/2),\end{cases}\qquad(1)$$

where $\exp_p^{-1}(x)=t\xi$, $\|\xi\|=1$. As a result, we obtain the following

Theorem 1 (the hat theorem). *Let s_1 and s_2 be local sections of a principal fiber bundle $P(M,G,\pi)$ and $s_1(p)=s_2(p)$. Then a sequence of neighborhoods $U\supset B(p;R)\supset B(p;R/2)\ni p$ and a local section $s:U\to P(M,G,\pi)$ exist such that $s=s_1$ on $U\setminus\overline{B}(p;R)$ and $s=s_2$ on $\overline{B}(p;R/2)$. Formula* (1) *provides an example of such a section.*

Further, let $P'(M,G',\pi)$ be a reduced subbundle of a bundle $P(M,G,\pi)$, $\{U_\alpha\}$ be an open covering of M with transition functions $\psi_{\beta\alpha}$ taking values in G'; assume that $U\ni p$ is a neighborhood from $\{U_\alpha\}$.

Consider the manifold $M'=M\setminus\overline{B}(p;R)$ with the corresponding open covering $\{U'_\alpha\}$, $U'_\alpha=U_\alpha\setminus\overline{B}(p;R)$ and transition functions $\psi'_{\beta\alpha}=\psi_{\beta\alpha|M'}$. Then open covering $\{U'_\alpha;U\}$ with the transition functions $\psi'_{\beta\alpha}$ and the section s over U given by (1) determines a new subbundle of $P(M,G,\pi)$. Thus, we obtain the following