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ITERATIVE REARRANGEMENTS OF FUNCTIONS AND THE LORENZ SPACES

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Let a function $f : R^n \to R$ be measurable in R^n, finite almost everywhere, and obey $\lambda_f(\sigma) = \mathrm{mes}_n\{x \in R^n : |f(x)| > \sigma\} < \infty$ for all $\sigma > 0$. By a nonincreasing rearrangement of the function $f(x)$ we shall call a function $f^*(t)$, which does not increase on $(0, +\infty)$ and is equimeasurable with $|f(x)|$. It can be given via the equality

$$f^*(t) = \inf\{\sigma : \lambda_f(\sigma) \le t\}, \quad t > 0.$$

By the Lorenz space $L^{p,q}(R^n)$ we shall call a space of all measurable on R^n functions, provided that

$$\|f\|^*_{L^{p,q}(R^n)} = \left(\frac{q}{p}\int_0^\infty [t^{1/p} f^*(t)]^q \frac{dt}{t}\right)^{1/q} < \infty, \quad 0 < p, q < \infty.$$

By a rearrangement of the function $f(x)$, $x = (x_1, x_2, \dots, x_n)$, with respect to the first variable we shall mean the function $\mathcal{R}_1 f(s_1, x_2, \dots, x_n)$, measurable in $R_+ \times R^{n-1}$, nonincreasing with respect to s_1, and such that the functions $\mathcal{R}_1 f(s_1, \cdot)$ and $f(x_1, \cdot)$ are equimeasurable as functions of one variable for almost all remaining variables (see [1]). In a similar way, by rearranging $\mathcal{R}_1 f(s_1, x_2, \dots, x_n)$ with respect to other variables, we obtain the function $\mathcal{R}_{1,2,\dots,n} f(s_1, s_2, \dots, s_n)$, which is equimeasurable with $f(x)$, does not increase with respect to every variable. We shall call it iterative rearrangement of the function $f(x)$. It is necessary to note that the order, in which we rearrange functions, is essential. For instance, $\mathcal{R}_{1,2,\dots,n} f \ne \mathcal{R}_{n,n-1,\dots,1} f$. By the Lorenz spaces $\mathcal{L}^{p,q}(R^n)$ and $\mathcal{L}^{p,q}_*(R^n)$ we shall call the spaces defined via the equalities

$$\|f\|^*_{\mathcal{L}^{p,q}(R^n)} = \| \cdots \| f\|^*_{L^{p,q}(R)} \cdots \|^*_{L^{p,q}(R)},$$
$$\|f\|^*_{\mathcal{L}^{p,q}_*(R^n)} = \| \cdots \| \mathcal{R}_{1,2,\dots,n} f\|^*_{L^{p,q}(R)} \cdots \|^*_{L^{p,q}(R)},$$

where the norm of the space $L^{p,q}(R)$ is taken consecutively with respect to each variable, starting with the first, while other ones are fixed.

In [2] it was proved that for $p \ne q$, $q \ne \infty$, none of the spaces $L^{p,q}(R^n)$, $\mathcal{L}^{p,q}(R^n)$ is a subset of another one. The main result of the present article in given in the following

Theorem 1. (i) *If* $0 < p < q < \infty$, *then* $L^{p,q}(R^n) \subset \mathcal{L}^{p,q}_*(R^n)$.
(ii) *If* $0 < q < p < \infty$, *then* $\mathcal{L}^{p,q}_*(R^n) \subset L^{p,q}(R^n)$.
(iii) *If* $p \ne q$, *then* $L^{p,q}(R^n) \ne \mathcal{L}^{p,q}_*(R^n)$.

Proof. For the sake of clarity, we shall prove the Theorem for $n = 2$. For $n > 2$, the proof is to be carried out in analogous way.

In what follows, we shall take advantage of the fact that functions, which do not increase with respect to every variable, are continuous a.e. Since we are not able to give here the necessary reference, we shall simply prove this fact.