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THE BIORTHOGONAL SYSTEMS, GENERATED BY CERTAIN INVOLUTARY OPERATORS

F.N. Garifianov

Introduction

It is well known, that the operator of the singular integration

$$(S\varphi)(t) \equiv \frac{1}{\pi i}\int_\Gamma (\tau - t)^{-1}\varphi(\tau)d\tau,$$

in the sense of the Cauchy's principal value, under some rather general assumptions concerning the density $\varphi(t)$ and the simple closed curve Γ, is the involutory operator. It is sufficient to require, for instance, that $\varphi(t) \in L_p(\Gamma)$, $p > 1$ (see [1], p. 13). The set of the fixed points of the operator is the set of all functions which are holomorphic inside Γ, and the solutions of the equation

$$S\varphi = -\varphi \tag{1}$$

are holomorphic outside Γ and vanish at the infinity.

In this paper, we consider the properties of the biorthogonal systems generated by the involutory operators whose kernels have more sophisticated structure than the Cauchy kernel. Let us suppose that R is a square with vertices $t_1 = -t_3 = -\frac{1}{2}(1+i)$ and $t_2 = -t_4 = \frac{1}{2}(1-i)$, numbered in the order of the positive path-tracing, and $\Gamma = \partial R$. In this paper, we investigate the integral operator

$$(S\varphi)(t) \equiv \frac{1}{\pi i}\int_\Gamma E(\tau - t)\varphi(\tau)d\tau \tag{2}$$

with the odd kernel function

$$E(z) = z^{-1} + (z+1+i)^{-1} + (z+1-i)^{-1} + (z-1+i)^{-1} + (z-1-i)^{-1}, \tag{3}$$

and the biorthogonally conjugate systems generated by this operator. Note, that operator (2) is a singular perturbed operator with fixed singularity (see the general theory, e. g., in [1], [2]).

In Section 1, we carry out the equivalent regularization of equation (1) and describe the set of its solutions. In Section 2, we introduce the system of functions

$$\{\varphi_m\} : S\varphi_m = -\varphi_m + q_m^+; \quad q_m(z) = (-1)^m (m!)^{-1}(z^m - p_m), \quad m = 1, 2, \ldots, \tag{4}$$

where p_m are certain constants determined below. We prove that it is biorthogonally conjugate on Γ with the system

$$\{E^{(k)}(\tau)\}, \quad k = 1, 2, \ldots, \tag{5}$$

in the sense of the following relations:

$$(\varphi_m, E^{(k)}) \equiv \frac{1}{2\pi i}\int_\Gamma \varphi_m(\tau)E^{(k)}(\tau)d\tau = \delta_{m,k}. \tag{6}$$

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