

INVESTIGATION OF SYSTEMS OF LINEAR DIOPHANTINE INEQUALITIES BY THE GROUP THEORY METHODS

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We consider the system of linear Diophantine inequalities

$$\langle \mathbf{a}^i, \mathbf{x} \rangle \leq b_i, \quad i = 1, \dots, l; \quad (1)$$

$$\mathbf{x} \in Z^n, \quad (2)$$

where Z^n is the set of integer-valued vectors of arithmetic space R^n , $\langle \mathbf{a}^i, \mathbf{x} \rangle$ is the scalar product of vectors \mathbf{a}^i and \mathbf{x} .

We suppose that the set of solutions of inequality system (1) is nonempty and bounded. Condition (2) is an invariant of the group of motions of integer lattice Z^n . Therefore, there is a reason for applying the methods of the group theory in investigating system (1), (2).

Let Λ^0 be the permutation group on the set $\{1, \dots, n\}$. We denote by $\mathbf{x}\lambda = (x_{\lambda(1)}, \dots, x_{\lambda(n)})$ the result of the action of the permutation $\lambda = (\lambda(1), \dots, \lambda(n))$ on the vector $\mathbf{x} = (x_1, \dots, x_n)$. In this sense, the group Λ^0 is a finite subgroup of the group of motions of an integer lattice and, at the same time, a finite subgroup of an orthogonal group.

First we symmetrize system (1). Put $C = \{\mathbf{a}^i\lambda \mid \forall \lambda \in \Lambda^0, i = 1, \dots, l\} = \{\mathbf{c}^1, \dots, \mathbf{c}^m\}$; $\mathbf{c}^i = \mathbf{a}^i$, $i = 1, \dots, l$. We determine the numbers d_i as follows. If $i = 1, \dots, l$, then $d_i = b_i$; if $i = l+1, \dots, m$, then $d_i = \max\{\langle \mathbf{c}^i, \mathbf{x} \rangle \mid \langle \mathbf{a}^k, \mathbf{x} \rangle \leq b_k, k = 1, \dots, l\}$. The system

$$\langle \mathbf{c}^i, \mathbf{x} \rangle \leq d_i, \quad i = 1, \dots, m, \quad (3)$$

is equivalent to system (1). By definition, the set C is symmetric with respect to the group Λ^0 , i.e., $C\Lambda^0 = \{\mathbf{c}^i\lambda \mid \forall \lambda \in \Lambda^0, i = 1, \dots, m\} = C$. We shall say that systems (3) and (2), (3) are symmetric with respect to the group Λ^0 . If a system is symmetric with respect to the group Λ^0 , it is the same with respect to any of its subgroups.

The action of the group Λ^0 permutes the vectors \mathbf{c}^i . To the permutation $\lambda \in \Lambda^0$ there corresponds a permutation $\pi = (\pi(1), \dots, \pi(m))$ such that $\mathbf{c}^i\lambda = \mathbf{c}^{\pi(i)}$. The set of these permutations forms the permutation group Π^0 over the set $\{1, \dots, m\}$, generated by the group Λ^0 and the set C .

Lemma 1. *The groups Λ^0 and Π^0 are isomorphic.*

Proof. Since the set of solutions of system (3) is nonempty and bounded, the set C contains n linearly independent vectors. Let these vectors be namely $\mathbf{c}^1, \dots, \mathbf{c}^n$. The mapping Λ^0 to Π^0 is a homomorphism. Let λ belong to the kernel of the homomorphism. By representing arbitrary vector $\mathbf{x} \in R^n$ via a linear combination of the vectors $\mathbf{c}^1, \dots, \mathbf{c}^n$, we obtain

$$\mathbf{x}\lambda = \left(\sum_{i=1}^n \alpha_i \mathbf{c}^i \right) \lambda = \sum_{i=1}^n \alpha_i (\mathbf{c}^i \lambda) = \sum_{i=1}^n \alpha_i \mathbf{c}^i = \mathbf{x}.$$

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