Russian Mathematics (Iz. VUZ) *Izvestiya VUZ. Matematika*
Vol. 50, No. 6, pp. 76–82, 2006 *UDC 517.55*

INVERSION OF THE L'HOSPITAL RULE FOR HOLOMORPHIC IN A BALL FUNCTIONS

A.V. Shishkina

The assertion inverse to the L'Hospital rule, in general, is not true. It means that the existence of a limit of quotients of functions does not necessarily imply the existence of a limit of quotients of derivatives of these functions. However, under certain additional assumptions, the L'Hospital rule is invertible. For example, let us adduce the Hardy and Littlewood theorem ([1], pp. 215–216). If a function $f(x)$ is differentiable on $(0,1)$, $f'(x)$ increases on $(0,1)$, $c>0$, and $\lim\limits_{x\to 1-} f(x)(1-x)^c = A > 0$, then $\lim\limits_{x\to 1-} f'(x)(1-x)^{c+1} = Ac$ exists.

In [2], one studies the invertibility of the L'Hospital rule for analytic functions. In particular, one proves the following assertion.

Theorem A. *Assume that functions $f(z)$ and $g(z)$ are analytic in Δ, $\eta\in(0,\pi/2)$, $A\in C$, W_η is a Stolz angle from Δ, whose value equals 2η and the vertex is located at the point $z=1$; let $\lim\limits_{W_\eta\ni z\to 1}\frac{f(z)}{g(z)} = A$ exist. If the values of the function $\frac{g'(z)}{g(z)}(1-z)$ are separated from zero in W_η for $z\to 1$ then $\lim\limits_{W_{\eta-\varepsilon}\ni z\to 1}\frac{f'(z)}{g'(z)} = A$ exists for any $\varepsilon\in(0,\eta)$. If $\lim\limits_{W_\eta\ni z\to 1}\frac{g'(z)}{g(z)}(1-z) = 0$ then for any $\varepsilon\in(0,\eta)$, $\lim\limits_{W_{\eta-\varepsilon}\ni z\to 1}\frac{f'(z)}{g(z)}(1-z) = 0$.*

In a particular case, when the function $g(z)=(1-z)^{-c}$, $c\in C$, Theorem A implies the result from [3].

One adduces multidimensional analogues of the problem under consideration in [4] and [5].

Let C^n be an n-dimensional complex space of vectors $z=(z_1,z_2,\dots,z_n)$, where $z_1,z_2,\dots,z_n\in C$; let $B^n=\{z:\|z\|<1\}$ be a Euclidean ball in the space C^n. If $\varepsilon>0$ then denote $B^n_\varepsilon=\varepsilon B^n$.

Studying many questions related to the boundary behavior of functions in a multidimensional case, one uses the Koranyi–Stein domain instead of the Stolz angle.

Let $\alpha>1$, $e_1=(1,0,\dots,0)\in C^n$, then the Koranyi–Stein domain $\Omega_\alpha=\Omega_\alpha^{e_1}$ with the vertex e_1 is the set of all $z\in B^n$ such that $|1-z_1|<\frac{\alpha}{2}(1-\|z\|^2)$ (e.g., [6]).

In this paper, we develop the investigation performed in [4] and [5]. We prove an analog of Theorem A for a multidimensional case. Denote by $\phi_r(z)$, $r\in(0,1)$, the automorphism of the ball B^n [6],

$$\phi_r(z) = (\phi_{r1}(z),\dots,\phi_{rn}(z)),$$

where

$$\phi_{r1}(z)=\frac{r-z_1}{1-rz_1},\quad \phi_{rk}(z)=-\frac{z_k\sqrt{1-r^2}}{1-rz_1},\quad 2\le k\le n,$$

and let for $\varepsilon>0$, $\Phi_\varepsilon=\bigcup\limits_{r\in(0,1)}\phi_r(B^n_\varepsilon)$.

The work was supported by the Competitive Center for Fundamental Science, grant no. A04-2.8-719.