Russian Mathematics (Iz. VUZ) *Izvestiya VUZ. Matematika*
Vol. 41, No. 7, pp. 21–27, 1997 *UDC 517.5*

EMBEDDING THEOREMS AND EXTERIOR CROSS MEASURES

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The generalized Sobolev spaces, analogous to classes $\overset{\circ}{W}{}^1_p(\Omega)$ and arising in replacement of $L_p(\Omega)$ by symmetric spaces, are studied. Exact with respect to the order estimates of the norm of embedding operators of spaces under investigation into spaces with mixed norm are established. The base of most proofs is the Lumis-Whitney inequality which relates the Lebesgue measure of a set with its cross measures.

1. Below $\mathbb{R}^n$ will stand for n-dimensional Euclidean space of points $x = (x_1, \dots, x_n)$, $|x| = \sqrt{x_1^2 + \dots + x_n^2}$; $\mathfrak{B}_n$ is the set of Borel subsets of the space $\mathbb{R}^n$; $\Pi_\tau : \mathbb{R}^n \to \tau$ is the operator of orthogonal projection of $\mathbb{R}^n$ to a proper subspace τ; $\tau_x^\perp$ is the space orthogonal to τ and passing through x from τ; T_n^m is the set of m-dimensional coordinate subspaces of the space $\mathbb{R}^n$ ($m = 1, \dots, n-1$); $\operatorname{mes}_n A$ means the n-dimensional Lebesgue measure of a set A.

Let us put into correspondence to a set A from $\mathfrak{B}_n$ and a space τ from T_n^m the number $\operatorname{mes}_m \Pi_\tau A$, which is called (see [1], p. 227) *exterior cross measure* of the set A. The most of C_n^m numbers $\operatorname{mes}_m \Pi_\tau A$ ($\tau \in T_n^m$) is denoted by $\mathcal{D}_m(A)$. The number $\mathcal{D}_m(A)$ characterizes the m-dimensional extent of the set A. Sometimes it is called *m-dimensional diameter* of the A. The Lumis-Whitney inequality (see [1], p.227; [2]) implies the estimate

$$\operatorname{mes}_n A \le \mathcal{D}_m^{n/m}(A) \quad (A \in \mathfrak{B}_n). \tag{1}$$

Further we shall use terminology and results of the theory of symmetric spaces (see [3]–[5]). Let $I = (0, b)$ $(0 < b \le \infty)$, $E = E(I)$ and $E_1 = E_1(I)$ be symmetric spaces of measurable with respect to the measure mes_1 functions, and $E \subset E_1$. We denote by E_1/E (see [4]) the set of measurable on I functions, for which there makes sense the norm:

$$\|u;\, E_1/E\| = \sup\{\|uv;\, E_1\|,\ \|v;\, E\| \le 1\}.$$

The space E_1/E is called the *space of multiplicators* from E to E_1. Definition of E_1/E implies the estimate

$$\|uv;\, E_1\| \le \|u;\, E_1/E\|\, \|v;\, E\|.$$

The space E_1/E is symmetric. In particular, the same is the space $E' = L_1/E$, which is said to be (see [3], p. 140) associated to the space E. The inequality $\|uv;\, L_1\| \le \|u;\, E'\|\, \|v;\, E\|$ is termed the Hölder inequality for symmetric spaces. A symmetric space E is said to be (see [3], p. 141) *maximal* if E is isometric to the second associated space E''.

By the known scheme (see [3], pp. 211–213) the space $E = E(I)$ and a Lebesgue measurable set $\Omega \subset \mathbb{R}^n$ $(0 < \operatorname{mes}_n \Omega \le b)$ generate the space $E(\Omega)$ of measurable with respect to the measure mes_n functions on Ω. In this situation, $\|f;\, E(\Omega)\| = \|f^*;\, E(I)\|$, where f^* is the rearrangement of the function $|f|$ in descending order (see [3], p. 83; [6], p. 332; [7]).