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On the Word Problem for the Free Burnside Semigroups Satisfying $x^2 = x^3$

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Abstract—We study the word problem for the free Burnside semigroups satisfying $x^2 = x^3$. For any $k > 2$, we reduce this problem for the k-generated free Burnside semigroup $B(2,1,k)$ to the word problem for the two-generated semigroup $B(2,1,2)$. Furthermore, if every element of $B(2,1,2)$ is a regular language, then every element of $B(2,1,k)$ appears to be a regular language as well. Thus, Brzozowski's conjecture holds for the semigroup $B(2,1,k)$ if and only if it holds for $B(2,1,2)$.

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1. Free Burnside semigroups are free objects in semigroup varieties $\mathbf{var}\{x^n = x^{n+m}\}$ with some positive integers n and m. Thus, a free Burnside semigroup satisfying the identity $x^n = x^{n+m}$ over a k-element alphabet Σ_k is defined as the quotient semigroup $B(n,m,k) = \Sigma_k^+/\sim_{n,m,k}$ of the free semigroup Σ_k^+ by the congruence $\sim_{n,m,k}$ generated by all pairs in the form (X^n, X^{n+m}), where $X \in \Sigma_k^+$.

Free Burnside semigroups were intensively studied by many authors. These semigroups are described in terms of both the semigroup theory and the formal language theory (note that elements of $B(n,m,k)$ are languages over Σ_k). See the survey [1] for the theory of Burnside semigroups.

The word problem is probably the most important problem related to free Burnside semigroups. For a semigroup $B(n,m,k)$ this problem is stated as follows: *Given words U and V over Σ_k, decide whether U and V represent the same element of $B(n,m,k)$.* Obviously, free Burnside semigroups $B(n,m,1)$ are cyclic and finite, and the word problem for $B(n,m,1)$ is trivially solvable. In what follows we assume that $k > 1$. According to results obtained by J. Green and D. Rees [2] and later by L. Kad'ourek and J. Polák [3], the word problem for the case $n = 1$ is reducible to the corresponding word problem for free Burnside groups (i.e., free groups in the variety $\mathbf{var}\{x^m = 1\}$).

For semigroups $B(n,m,k)$ with $n \geq 2$ the word problem is closely related to the Brzozowski conjecture which claims that every element of $B(n,m,k)$ is a regular language. Since for any regular language there exists a finite automaton recognizing this language, the Brzozowski conjecture implies the solvability of the word problem. In papers [4–8] the Brzozowski conjecture is confirmed for the case $n \geq 3$. Hence, the word problem is solvable in this case. For semigroups $B(2,m,k)$ the word problem is still open. Moreover, with $m \geq 2$ such semigroups contain elements which are not regular languages [9]. Therefore, the Brzozowski conjecture is not valid when $n = 2$ and $m \geq 2$. For semigroups $B(2,1,k)$ the Brzozowski conjecture is neither proved nor disproved yet. Note that this particular case was explicitly mentioned by J. Brzozowski [10].

The following theorems represent the main results of this paper.

Theorem 1. *The word problem for a semigroup $B(2,1,k)$ with $k > 2$ is solvable if and only if it is solvable for the semigroup $B(2,1,2)$.*

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