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# OPERADS RELATED TO THE OPERAD OF SYMMETRIC GROUPS. I

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In the first part of this paper, we construct and study an infinite family of operads (linear and nonlinear) containing many known operads which were early described separately. These are, in particular, the operads whose components are symmetric or hyperoctahedral groups. Although we generalize some well-known objects, it is difficult to point out the literature where similar investigations have been carried out even for the operad of symmetric groups. We also describe the varieties of algebras over operads from the family under study. Among such varieties are the varieties of semigroups and associative algebras, as well as the varieties of semigroups and associative algebras with involution.

In the second part of the paper, we study the operad defined on the set of Young diagrams and some related operads. We describe the varieties of algebras over these operads. The notation and terminology in this paper mainly follow [1].

## 1. Permutations and partitions

Let $n$ be a nonnegative integer. In this paper, $[n]$ denotes either the set $\{1,\dots,n\}$ or the set $\{0,1,\dots,n\}$. The results of the Section 1 hold in both the cases. By a partition of the set $[n]$ into $m$ $(m>0)$ parts we mean an ordered sequence of positive integers (in the case $[n]=\{1,\dots,n\}$) or that of nonnegative integers (in the case $[n]=\{0,1,\dots,n\}$) $\alpha=(n_1\dots n_m)$ such that $|\alpha|=n_1+\dots+n_m=n$. The set of all partitions of $n$ into $m$ parts is denoted by $P(n,m)$.

We define the composition $P(n,m)\times P(m,k)\to P(n,k)$ as follows. Let $\alpha\in P(n,m)$, $\beta\in P(m,k)$, $\beta=(m_1\dots m_k)$, $m_1+\dots+m_k=m$, and let $\alpha=(n_{11}\dots n_{m_11}\dots n_{1k}\dots n_{m_kk})$. Then we define $\alpha\beta\in P(n,k)$ to be $\Big(\sum\limits_{i=1}^{m_1} n_{i1}\dots\sum\limits_{i=1}^{m_k} n_{ik}\Big)$.

**Lemma 1.** *With respect to the above defined composition operation, $P$ is a category with objects $[n]$, $n=1,2,\dots$ (or $n=0,1,2,\dots$).*

**Proof.** One can immediately verify that the composition defined above is associative. The identity morphism $1_{[n]}$ is the partition $\underbrace{(1\ 1\dots 1)}_{n}=1^n$. $\square$

There is a functor $P\times P\xrightarrow{\circ}P$ with the property of strict associativity. $[n]\circ[k]=[n+k]$ and if $\alpha=(n_1\dots n_m)$, $\beta=(k_1\dots k_l)$, then $\alpha\circ\beta=(n_1\dots n_m\ k_1\dots k_l)$. This means that the following identities hold: $([n]\circ[k])\circ[m]=[n]\circ([k]\circ[m])$, $(\alpha\circ\beta)\circ\gamma=\alpha\circ(\beta\circ\gamma)$ and $(\alpha\gamma)\circ\beta=(\alpha\circ\beta)(\gamma\circ 1_{[q]})=(\alpha\circ 1_{[p]})(\gamma\circ\beta)$, where $[n]\xrightarrow{\alpha}[m]\xrightarrow{\gamma}[k]$, $[p]\xrightarrow{\beta}[q]$.

In what follows we will write $\alpha\beta$ instead of $\alpha\circ\beta$ because in each case under consideration it will be clear from the context what is meant: the composition of morphisms in the category $P$ or the functor $\circ$.

On $P(n,m)$, the following right action of the group $\Sigma_m$ of permutations of $m$-th degree is defined: if $\alpha=(n_1\dots n_m)$, $\sigma\in\Sigma_m$, then $\alpha\sigma=(n_{\sigma1}\dots n_{\sigma m})\in P(n,m)$. (We write $\sigma i$ instead of $\sigma(i)$.) In

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