ISSN 1066-369X, Russian Mathematics (Iz. VUZ), 2013, Vol. 57, No. 9, pp. 22–30. © Allerton Press, Inc., 2013.
Original Russian Text © M.A. Komarov, 2013, published in Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika, 2013, No. 9, pp. 28–37.

An Example of Non-Uniqueness of a Simple Partial Fraction of the Best Uniform Approximation

M. A. Komarov[1*]

[1] *Vladimir State University, ul. Gor'kogo 87, Vladimir, 600000 Russia*

Received June 19, 2012

Abstract—For arbitrary natural $n \geq 2$ we construct an example of a real continuous function, for which there exists more than one simple partial fraction of order $\leq n$ of the best uniform approximation on a segment of the real axis. We prove that even the Chebyshev alternance consisting of $n+1$ points does not guarantee the uniqueness of the best approximation fraction. The obtained results are generalizations of known non-uniqueness examples constructed for $n = 2, 3$ in the case of simple partial fractions of an arbitrary order n.

DOI: 10.3103/S1066369X13090041

Keywords and phrases: *simple partial fraction, approximation, uniqueness, alternance.*

1. Introduction. Recently, approximative properties of *simple partial fractions* (s. p. f.), i.e., logarithmic derivatives of algebraic polynomials

$$Q_n'/Q_n, \quad Q_n(z) := z^n + q_1 z^{n-1} + \cdots + q_{n-1} z + q_n, \quad z, q_k \in \mathbb{C}, \quad n \in \mathbb{N},$$

(n is the order of the fraction) became an object of active research. The first result on the uniform approximation of functions by simple partial fractions was obtained in [1], where it was shown that the class of functions approximable by s. p. f. in the uniform metric on a bounded set contains polynomials and, consequently, functions approximable by them. This fact implies an analog of the Mergelyan theorem on the polynomial approximation. It turned out that the rate of the approximation by s. p. f. for a wide class of functions and bounded sets has the same order as for the polynomial approximation. This allows one to obtain analogs of classical theorems by D. Jackson, S. N. Bernstein, A. Zigmund, V. K. Dzyadyk, and J. L. Walsh [2, 3]. There exist partial analogies with the classical Chebyshev alternance theorem; for example, in terms of the alternance[1)] one has obtained a sufficient condition for the best approximation of real-valued functions on segments of the real axis [4], which also is a sufficient condition for its uniqueness [5].

An important distinction of s. p. f. from polynomials is the possibility of approximation on unbounded sets. Recent papers by P. A. Borodin and O. N. Kosukhin [6], V. Yu. Protasov [7], V. I. Danchenko [8], and I. R. Kayumov [9] are devoted to this issue. It is established, in particular, that every function continuous on $\mathbb{R}$ and vanishing at infinity is arbitrarily exactly approximable by s. p. f. in the uniform metric [6]. At the same time, in spaces $L_p(\mathbb{R})$ with finite $p > 1$ the class of well-approximable functions narrows sharply [7]; it consists exactly of functions representable by convergent in $L_p(\mathbb{R})$ series $\sum_k (x - z_k)^{-1}$. Some convergence criteria for such series are obtained in [8, 9].

Another important distinction is the phenomenon of the non-uniqueness of s. p. f. of the best approximation. The first example of the non-uniqueness with $n = 2$ was published by V. I. Danchenklo and E. N. Kondakova [10]; namely, for the function $f(x) = x + 1$ there exists infinitely many s. p. f. of the best approximation on $[-1, 1]$ with order ≤ 2. There arises the question on constructing a non-uniqueness example for arbitrary $n > 2$. For $n = 3$ such example was constructed by the author in [11].

*E-mail: kami9@yandex.ru.

1)Let us recall the definition of the alternance. Let R_n be a real-valued s. p. f. without poles on a segment $I \subset \mathbb{R}$. It is said that points $z_1 < z_2 < \cdots < z_s$ from the segment I form a (*Chebyshev*) *alternance* for the difference $R_n - f$, if $R_n(z_k) - f(z_k) = \pm(-1)^k \|R_n - f\|_{C(I)}$, $k = \overline{1, s}$.