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A MODIFIED LAGRANGE FUNCTION FOR THE LINEAR PROGRAMMING PROBLEMS

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1. For a modified Lagrange function (MLF) in the linear programming (LP) problem we determine the exact value of the penalty coefficient, starting from which the solution of the LP problem is found as a result of a single minimization of a convex quadratic function on the positive orthant.

Consider the primal and the dual linear programming problems:

$$f_* = \min_{x \in X} c^\top x, \quad X = \{x \in R^n_+ : b - Ax = 0_m\}, \tag{1}$$

$$\max_{u \in U} b^\top u, \quad U = \{u \in R^m : c - A^\top u \geq 0_n\}. \tag{2}$$

Here and in what follows A is an $m \times n$-matrix of the rank m, $m < n$, $d = n - m$ is its deficiency, $c \in R^n$, $b \in R^m$, $x \in R^n_+$, $u \in R^m$ are vectors, we denote by 0_i the zero i-dimensional vector.

To solve the LP problem we use a modified Lagrange function

$$H(x, u, \varepsilon) = c^\top x + u^\top (b - Ax) + \|b - Ax\|^2 / 2\varepsilon,$$

where $\varepsilon > 0$ is the penalty coefficient, $\|a\|$ is the Euclidean norm of a in the relevant space.

Consider the auxiliary problem:

$$\min_{x \in R^n_+} H(x, \tilde{u}, \varepsilon) \tag{3}$$

with a fixed vector $\tilde{u}$ of the Lagrange multipliers. If $\tilde{u} \equiv 0_m$, then problem (3) is related to the method of the exterior quadratic penalty functions. The results presented below can be obviously extended also to this case.

Assume that the primal LP problem (1) has the unique (perhaps, singular) solution x_*. In this solution $x^L_* > 0_l$ is the totality of the positive coordinates, $l < m$. For the nonsingular solution we have $l = m$. We denote by I^L_* the set of indices corresponding to the positive components of x_*. If x_* is a singular solution, then the dual LP problem (2) has a non-unique solution. From the solutions set U_* of the dual problem (2) select the solution u_*, which is the closest in the Euclidean norm to a certain given vector $\tilde{u}$. Therefore, u_* is the solution of the problem

$$\min\{\|u - \tilde{u}\|^2/2 : c - A^\top u \geq 0_n,\ b^\top u \geq f_*\}, \tag{4}$$

and at $[x_*, u_*]$ the Kuhn-Tucker conditions for problem (1) hold, which can be rewritten in more detail as

$$\begin{aligned} &v^L_* = c^L - B^\top_L u_* = 0_l, \quad x^L_* > 0_l; \quad v^S_* = c^S - B^\top_S u_* = 0_s, \quad x^S_* = 0_s; \\ &v^N_* = c^N - N^\top u_* > 0_d, \quad x^N_* = 0_d; \qquad B_L x^L_* = b. \end{aligned} \tag{5}$$

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