ISSN 1066-369X, Russian Mathematics (Iz. VUZ), 2010, Vol. 54, No. 7, pp. 49–55. © Allerton Press, Inc., 2010.
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Curvature Identities for Principle T^1-Bundles Over Almost Hermitian Manifolds

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Received July 25, 2008

Abstract—We show that the identities R_1, R_2 and R_3 for an almost Hermitian structure S on the base of the canonical principal T^1-bundle are equivalent to their contact analogs for the induced almost contact metric structure $S^\sharp$ on the total space of this bundle. We prove that the canonical connection of the canonical principal T^1-bundle over a Hermitian or a quasi-Kähler manifold of class R_3 is normal. We also prove that that the canonical connection of the canonical principal T^1-bundle over a Vaisman–Gray manifold M of class R_3 is normal if and only if the Lee vector of M belongs to the center of the adjoint K-algebra.

DOI: 10.3103/S1066369X10070054

Key words and phrases: *principal toroidal fiber bundle, almost contact structure, curvature tensor.*

The principal T^1-bundle is a geometric structure that makes it possible to establish fundamental connections between Hermitian and contact geometries. For example, it is well-known that, for an even-dimensional manifold M with almost Hermitian structure $S = (g, J)$, the canonical principal T^1-bundle is defined whose characteristic class is generated by the generalized Ricci form computed explicitly in [1]. The canonical connection form on this principal fiber bundle generates an almost contact metric structure $S^\sharp$ on the total space. In [1], a series of connections between these structures has been established. For example, it has been proved [1] that $S^\sharp$ is a quasi-Sasakian structure if and only if S is a Kähler structure, $S^\sharp$ is a Sasakian structure if and only if S is a Hodge structure, $S^\sharp$ is a cosymplectic structure if and only if S is a Ricci-flat Kähler manifold, etc.

On the other hand, in [2], A. Gray remarked that the Riemann–Christoffel tensor R of a manifold with fixed almost Hermitian structure $S = (g, J)$, in addition to the four classical symmetries, can possess possible additional symmetry properties, he singled out the three classes of almost Hermitian manifolds characterized by the identities:

$$R_1 : R(X, Y, Z, W) = R(JX, JY, Z, W);$$
$$R_2 : R(X, Y, Z, W) = R(JX, JY, Z, W) + R(JX, Y, JZ, W) + R(JX, Y, Z, JW);$$
$$R_3 : R(X, Y, Z, W) = R(JX, JY, JZ, JW).$$

In [2], it has also been established that $R_1 \subset R_2 \subset R_3$ and that, in the class of Kähler manifolds, all the three identities may take place, in the class of nearly Kähler manifolds, the identities R_2 and R_3 may take place, and, in the class of Hermitian manifolds, the identities R_2 and R_3 are equivalent.

In this paper we establish equivalence of the identities R_1, R_2, and R_3 for an almost Hermitian structure S on the base of the canonical principal T^1-bundle to their contact analogs CR_1, CR_2, and CR_3 respectively for the induced almost contact metric structure $S^\sharp$ on the total space of this bundle. We prove that the canonical connection of the canonical principal T^1-bundle over an Hermitian or a quasi-Kähler manifold of class R_3 is normal. We also prove that the canonical connection of the canonical principal T^1-bundle over a Vaisman–Gray manifold M of class R_3 is normal if and only if the Lee vector of M belongs to the center of the adjoint K-algebra.

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