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ON $(1+\varepsilon_n)$-BOUNDED M-BASES

R.V. Vershinin

Let us recall certain definitions from [1]. X stands for arbitrary separable Banach space. A sequence $\{x_n\}_1^\infty \subset X$ is called *minimal* if $x_m \notin [x_n]_{n\neq m}$, $m = 1, 2, \dots$ (we denote by $[x_n]$ a closure of linear span of $\{x_n\}$). The sequence $\{x_n\}$ is minimal if and only if there exists the sequence $\{x_n^*\}_1^\infty \subset ([x_n])^*$, such that the system $\{x_n, x_n^*\}_1^\infty$ is biorthogonal, i.e., $x_m^*(x_n) = \delta_{m,n}$, $m, n = 1, 2, \dots$ A minimal sequence $\{x_n\}$ is called *M-base* if it is complete in X and $\{x_n^*\}$ is total, i.e., $([x_n^*])^\top = 0$. M-base $\{x_n\}$ is called *norming* if $r([x_n^*]) > 0$ for Dixmier characteristics

$$r(V) = \inf_{x\in X} \sup_{x^*\in V} \frac{|x^*(x)|}{\|x^*\|\,\|x\|}, \quad V \subset X^*.$$

A biorthogonal system $\{x_n, x_n^*\}$ (or minimal sequence $\{x_n\}$) is called *C-bounded* if $\|x_n\|\,\|x_n^*\| < C$, $n = 1, 2, \dots$, and *normal* if $\|x_n\|\,\|x_n^*\| = 1$, $n = 1, 2, \dots$ We call a system $\{x_n, x_n^*\}$ *$(1+\varepsilon_n)$-bounded* if $\|x_n\|\,\|x_n^*\| < 1+\varepsilon_n$, $n = 1, 2, \dots$

A question on existence of bounded M-base in arbitrary separable Banach space X goes back to Banach. Davis and Johnson (see [2]) constructed complete $(1+\varepsilon)$-bounded minimal system in X for arbitrary $\varepsilon > 0$. The question was resolved finally in [3], where $(\sqrt{2}+1+\varepsilon)$-bounded M-base in X was obtained in another terms. But the known problem on existence of normal M-base (or, at least, normal complete minimal system) in X (see [1], Chap. III, § 8, p. 251) remains open. The answers to the following questions are also unknown.

Problem 1 (existence). Let $\{\varepsilon_n\}$ be positive values. Does X contain $(1+\varepsilon_n)$-bounded M-base and complete $(1+\varepsilon_n)$-bounded minimal system?

Problem 2 (extendibility). Let $\{\varepsilon_n\}$ be positive values, and let $\{x_n, x_n^*\}_1^N$ be finite $(1+\varepsilon_n)$-bounded biorthogonal system. Is it extendible to $(1+\varepsilon_n)$-bounded M-base and to complete $(1+\varepsilon_n)$-bounded minimal system?

Clearly, the positive solution of Problem 1 implies positive solution of Problem 2. The following steps in solving the problem are already done. A modification of considerations of [3] enabled to construct $(1+\varepsilon_n)$-bounded M-base in X (see [4]) and to obtain $(1+\varepsilon_n)$-bounded complete minimal system for certain sequences $\{\varepsilon_n\}$ such that $\varepsilon_n \to 0$ for $n \to \infty$ (see [5]).

In the present article the problems of existence and extendibility are resolved for arbitrary $\varepsilon_n > 0$ with $\sum \varepsilon_n^2 = \infty$. We modify the Pelczynski method to this end. But any method, where the elements of extension are chosen near the given subspace, cannot give us solution of the extendibility problem for fast decreasing ε_n.

First we reduce Problem 2 to construction of extension containing a given vector.

Lemma 1. *Let $\{\varepsilon_n\}_1^\infty$ be positive values such that for any vector $x \in X$ any finite $(1+\varepsilon_n)$-bounded biorthogonal system $\{x_n, x_n^*\}_1^N$ is extensible up to complete $(1+\varepsilon_n)$-bounded biorthogonal*