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A Generalization of the Helly Theorem for Functions with Values in a Uniform Space

Yu. V. Tret'yachenko[1*]

[1] *State University of Higher School of Economics in Nizhni Novgorod, ul. Bol'shaya Pecherskaya 23/12, Nizhni Novgorod, 603155 Russia*

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Abstract—In this paper we consider sequences of functions that are defined on a subset of the real line and take on values in a uniform Hausdorff space. For such sequences we obtain a sufficient condition for the existence of pointwise convergent subsequences. We prove that this generalization of the Helly theorem includes many results of the recent research. In addition, we prove that the sufficient condition is also necessary for uniformly convergent sequences of functions. We also obtain a representation for regular functions whose values belong to the uniform space.

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1. INTRODUCTION

According to the classical Helly selection principle [1] ([2], P. 208, lemma 2), each infinite uniformly bounded family of monotone real functions defined on a segment $[a, b]$ from $\mathbb{R}$ contains a pointwise convergent on $[a, b]$ subsequence. This Helly theorem is also valid for an arbitrary nonempty subset T from $\mathbb{R}$ (see, e.g., [3]) and for a uniformly bounded sequence of functions with uniformly bounded Jordan variations. Most generalizations of the Helly selection principle are based on the uniform boundedness of a sequence of functions and their generalized variations (see [4–7] for real-valued functions and [3, 8–16] for functions that take on values in a metric or Banach space). Such selection principles have many applications [3, 8–10, 12–14], because they are efficient in proving the existence theorems; for example, they are widely used in the convergence theory for Fourier series and in the theory of stochastic processes. Generalizations of the Helly theorems are also applied in the multivalued analysis for proving the existence of regular selections for multifunctions of bounded generalized variation, and in studying nonlinear superposition operators [3].

In [17, 18] one first presents a selection principle for one-variable functions with values in a uniform space. This principle implies most known generalizations of the Helly theorem with restrictions on generalized variations [3–16]. Moreover, the restriction on the *modulus of variation* of functions from the initial sequence that is a base of the selection principle proposed in [17, 18] is not only a sufficient condition for the existence of a pointwise convergent subsequence, but also the necessary condition for the uniform convergence of the sequence of functions (as distinct from the known selection principles [3–16]).

In this paper we generalize the Helly theorem for a sequence of functions with values in a Hausdorff uniform space; the obtained result includes the selection principle proposed in [17, 18].

Section 2 contains the main definitions and statements of the obtained results. In Section 3 we adduce some auxiliary assertions and study regular functions with respect to a dense subset of the definition domain. In Section 4 we prove the basic theorems and compare the main result (Theorem 1) with the selection principle proposed in [17, 18].

Results of this paper were announced in [19, 20].

*E-mail: tretyachenko_y_v@mail.ru.