ISSN 1066-369X, Russian Mathematics (Iz. VUZ), 2007, Vol. 51, No. 9, pp. 37–44. © Allerton Press, Inc., 2007.
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Geodesic Rigidity of Certain Classes of Almost Contact Metric Manifolds

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Received January 29, 2007

DOI: 10.3103/S1066369X07090022

Introduction. Paper [1] was a natural continuation of investigation of basic problems of the theory of geodesic mappings of pseudo-Riemannian spaces. This theory was originated in the end of the 19th century by T. Levi-Civita, T. Tomas, and H. Weyl. Since then many mathematicians worked in this area of research, studying geodesic mappings of pseudo-Riemannian manifolds endowed with additional structures. The theory was enriched by new results. Thus, in the middle of the 20th century, Westlake and Yano proved that Kähler manifolds admit no nontrivial geodesic transformations preserving the complex structure. Recently a contact analog of these results has been obtained. In particular, in [1] (p. 215), the notion of a contact-geodesic transformation of an almost contact metric structure was introduced as a geodesic transformation preserving the almost contact structure. In the same paper, a technique was developed with the use of which it was proved that cosymplectic and Sasakian structures, as well as Kenmotsu structures, admit no nontrivial contact-geodesic transformations of the metric.

In this paper, using the results of the above mentioned research, we prove that quasi-Sasakian structures admit no nontrivial contact-geodesic transformations of the metric, which, in turn, generalizes results of [1]. We also prove that regular locally conformally quasi-Sasakian structures admitting nontrivial contact-geodesic transformations of the metric are normal.

1. Preliminaries. Let M be a $(2n+1)$-dimensional smooth manifold, and let $\mathrm{X}(M)$ denote the module of smooth vector fields on M. Recall that (see [2], p. 446) an *almost contact structure* on a manifold M is a triple (η, ξ, Φ) of tensor fields on M, where η is a differential 1-form, called the contact form of the structure; ξ is a vector field, called the characteristic vector field; Φ is a tensor field of type $(1;1)$, called the structure endomorphism of the module $\mathrm{X}(M)$. These tensor fields satisfy the relations

$$1)\ \eta(\xi)=1; \quad 2)\ \eta\circ\Phi=0; \quad 3)\ \Phi(\xi)=0; \quad 4)\ \Phi^2=-\mathrm{id}+\eta\otimes\xi. \tag{1}$$

If, in addition, M is endowed with a Riemannian structure $g=\langle\cdot,\cdot\rangle$ such that $\langle\Phi X,\Phi Y\rangle=\langle X,Y\rangle-\eta(X)\eta(Y)$, $X,Y\in\mathrm{X}(M)$, then the set (η,ξ,Φ,g) is called an *almost contact metric* (briefly, AC-) *structure*. The tensor $\Omega(X,Y)=\langle X,\Phi Y\rangle$ is skew-symmetric and is called the *fundamental form of the AC-structure*. A manifold with fixed almost contact metric structure is called an almost contact metric manifold.

An AC-structure is called a *contact metric structure* if $d\eta(X,Y)=\langle X,\Phi Y\rangle$. An AC-structure is called *normal* if the Nijenhuis tensor of its structure endomorphism satisfies the identity $2N_\Phi+d\eta\otimes\xi=0$, where

$$N_\Phi(X,Y)=\frac{1}{4}([\Phi X,\Phi Y]+\Phi^2[X,Y]-\Phi[\Phi X,Y]-\Phi[X,\Phi Y]).$$

A normal contact metric structure is called a *Sasakian structure*. A normal almost contact metric structure is called a *quasi-Sasakian* (briefly, a QS-) *structure* if its fundamental form $\Omega(X,Y)=\langle X,\Phi Y\rangle$ is closed.

In the $C^\infty(M)$-module $\mathrm{X}(M)$ of smooth vector fields on an AC-manifold, there are intrinsically defined the two mutually complementary projectors $\mathrm{l}=-\Phi^2$ and $\mathrm{m}=\eta\otimes\xi=\mathrm{id}+\Phi^2$ onto the distributions $\mathrm{L}=\mathrm{Im}\,\Phi=\ker\eta$ and $\mathrm{M}=\ker\Phi$ of dimensions $2n$ and 1, respectively. Thus, $\mathrm{X}=\mathrm{L}\oplus\mathrm{M}$, $\mathrm{L}\cap\mathrm{M}=\{0\}$.