GEOMETRY OF HOMOGENEOUS Φ-SPACES OF ORDER 5

Yu.D. Churbanov

Introduction

The homogeneous periodical Φ-spaces (the Φ-spaces of order n, see Section 1) were first introduced by N.A. Stepanov, and were actively studied over the years, because, on the one hand, this class of spaces is the most natural direct generalization of the class of symmetric spaces; on the other hand, all the periodical homogeneous Φ-spaces are reductive (see [1]) and this is important from the geometrical point of view. Moreover, it was discovered that on the periodical Φ-spaces many invariant affinor structures exist compatible with symmetry of these spaces, whose operators on the tangent space can be given explicitly (see [2]).

In this article we study invariant classical affinor structures (almost product structures, almost complex structures, f-structures) of homogeneous Φ-spaces of order 5.

These spaces were first considered in [3], where the existence of almost complex structures was proved and the operators of these structures on the tangent space were found explicitly. Afterwards, in [4], [5], one structure among the invariant almost complex structures on a Φ-space of order 5 with the compact Lie group was studied; however, the operator of this structure was constructed by another method (see also [6], p. 107). Then, in [7], an invariant almost product structure of Φ-space of order 5 was constructed and studied; this made it possible to construct and study the same invariant almost complex structure from a new standpoint.

Next, in [2], the algebra of all invariant classical affinor structures on homogeneous regular Φ-spaces was described, for a periodical homogeneous Φ-space computational formulas for operators of these structures were given, and the cases $n = 3, 4, 5$ were considered in detail. In that paper it was also proved that on a homogeneous Φ-space of order 5 exactly one invariant almost product structure, two invariant f-structures, and two invariant almost complex structures exist.

Though the existence of almost complex structures on a homogeneous Φ-space of order 5 follows from the results in [2], in this article we construct these structures (see Section 3) by the original method from [7], with the use of an almost product structure considered in Section 2. For the sake of convenience, we define invariant f-structures (Section 4) with the use of almost complex structures and almost product structures; clearly, our construction agrees with the results obtained in [2].

We investigate the integrability problem for invariant almost complex structures (Section 3) and conclude that the class of homogeneous Φ-spaces G/H of order 5 with at least one integrable canonical almost complex structure is a subclass of local Φ-spaces of order 3 (see [8]), and both of the almost complex structures are integrable if and only if G/H is locally symmetric. We have a similar situation for the classical f-structures on homogeneous Φ-spaces of order 5.

In Section 5 we establish a relationship between invariant affine connections and canonical affinor structures on homogeneous Φ-spaces of order 5. In Section 7 we study naturally reductive homogeneous Φ-spaces of order 5. The class of these spaces is rather wide. In the case where