Russian Mathematics (Iz. VUZ) *Izvestiya VUZ. Matematika*
Vol. 43, No. 8, pp. 9–16, 1999 *UDC 517.948.32 : 517.544*

THE DOUBLY-PERIODIC MEROMORPHIC ANALOG OF THE CAUCHY KERNEL AND SOME ITS APPLICATIONS

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A sufficient number of works is devoted to the investigation of and solving conjugacy problems and boundary value problems for doubly-periodic piecewise analytic functions (see [1]–[7]). In all these works, in the capacity of an analog of the Cauchy kernel $\frac{1}{\tau-z}$ they take either the Weierstrass zeta-function $\zeta(\tau-z)$, or its modifications, which however are not doubly-periodic (and are "quasi-periodic", see [7]). The existence of a meromorphic analog of the Cauchy kernel on a Riemann surface (in particular, of the doubly-periodic kernel) was established in [8], where its explicit expression via the principal functionals of Riemann surface was cited. In the present article with the use of this fact and the theory of analytic functions (see [9]) we give a new analytic expression for the doubly-periodic Cauchy kernel and suggest some its applications.

We shall consider doubly-periodic (elliptic) functions with the principal periods Ω, Ω' (see [9], p. 9), where $\operatorname{Im}\frac{\Omega'}{\Omega}>0$. The domain of definition of any doubly-periodic function $f:\widehat{\mathbb{C}}/(Z\Omega+Z\Omega')\to\widehat{\mathbb{C}}$ is homeomorphic to a torus, therefore the theory of elliptic functions is sometimes called the *theory of functions on torus* (see [10], p. 42). It is convenient to represent this domain of definition in the form of a "fundamental parallelogram". We shall use this term for a parallelogram constructed on the vectors Ω, Ω', which are brought to a common origin; moreover, we agree to include in it: all interior points, two (i. e., the vectors Ω, Ω') of the four lateral sides, one (i. e., the common origin of the vectors Ω, Ω') of the four vertices. We denote by c this vertex, by Π the parallelogram proper. The fundamental parallelogram Π is convenient by the following feature: to its points the points of the Riemann surface $\mathbb{C}/(Z\Omega+Z\Omega')$ correspond in a bijective way.

For example, let for the points $q',q''\in\mathbb{C}$ the comparison be valid $q''\equiv q'$ modulo the periods Ω, Ω'. By the definition (see [9], p. 14), this means that $(q''-q')\in Z\Omega+Z\Omega'$. We shall write this as follows $q''\equiv q'(\operatorname{mod}(\Omega,\Omega'))$. If we assume that $q',q''\in\Pi$, then this comparison turns into the equality $q''=q'$.

To construct a doubly-periodic meromorphic analog of the Cauchy kernel, we shall need the Weierstrass sigma-function (see [9], p. 54), which corresponds to the lattice $Z\Omega+Z\Omega'$; the latter can be represented in the form of the infinite product (see [9], p. 55)

$$\sigma(q)=q\Pi'(1-q/s)e^{\frac{q}{s}+\frac{q^2}{2s^2}}\quad(s=\Omega m+\Omega' m'). \tag{1}$$

Here $'$ means that the product is extened to all integer m, m', except for the pair $m=m'=0$.

In the theory of elliptic functions (see [9]), the question on the existence of an elliptic function $f(q)$ with zeros at the points $a_1,\dots,a_n\in\Pi$ and the poles at the points $b_1,\dots,b_n\in\Pi$, where each zero and each pole is written in succession as many times as its multiplicity is and in Π the function $f(q)$ must have neither any other zero, nor pole, was completely resolved. If the