Russian Mathematics (Iz. VUZ) *Izvestiya VUZ. Matematika*
Vol. 41, No. 9, pp.57–70, 1997 *UDC 514.76*

GEOMETRY OF THE GRASSMANN BUNDLE. I

N.K. Farafonova

Introduction

Investigation of the geometry of total space of bundle has assumed a significance of its own after works by B.L. Laptev and Sasaki, appeared in the fifties (see [1] and [2] for a historical account). The aim of this article is to establish connection between the geometrical properties of a Grassmann bundle and its base which is supposed to be a Riemannian manifold. We endow the Grassmann bundle with a natural Riemannian metric (see [6] for its geometrical sense). A similar problem for tangent bundle, spherical bundle, and normal bundle has been investigated by Sasaki, Dombrovskii, Mok, Yampol'skiĭ et al. For a detailed information on the results concerning this problem the reader can be referred to [2], § 3. In the first part of the present article we construct a generating system of vector fields on the Grassmann bundle whose properties allow us to connect the geometry of base manifold with the geometry of total space. The central problem with the Grassmann bundle is to construct special vector fields on the total space, which are lifts of skew-symmetric tensor fields (see Definition 2.2).

In what follows the Riemannian manifolds are assumed to be differentiable of class C^∞ (both the manifolds and their Riemannian metrics). We shall use the Einstein summation convention.

1. Skew-symmetric tensors on Riemannian manifold

In this Section we give a brief account of notions and properties (both known and specially introduced here) for (1,1)-tensors on a Riemannian manifold M with a Riemannian metric tensor g, which are used in Section 3 as technical tools in the investigation of the Grassmann bundle geometry. For our purposes the most suitable thing is to understand a tensor field of type (1,1) as a linear mapping of the module of smooth vector fields on M into itself ([3], Chap. 1, Example 3.1). Then a (1,1)-tensor at a point $u_0 = (u_0^a)$ of M is the following linear operator

$$Q = Q_a^b du^a \otimes \frac{\partial}{\partial u^b} : T_{u_0}M \to T_{u_0}M \tag{1}$$

in the tangent space. The components $Q_a^b(u^c)$ of Q with respect to a local coordinate system u^b are smooth functions. A (1,1)-tensor field Q on M is said to be skew-symmetric if Q is skew-symmetric at each point with respect to the scalar product induced by the Riemannian metric g, i. e.,

$$g(QX, Y) + g(X, QY) = 0 \tag{2}$$

for all tangent vectors X, Y. In what follows, except otherwise specially stipulated, skew-symmetric tensor fields of type (1,1) are called skew-symmetric tensor fields, for short. In case where from the context it is clear with respect to what basis the matrix of a linear operator or a quadratic form is