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$\mathbb{R}$-Conformal Invariants of Curves

I. S. Strel'tsova[1*]
(Submitted by V.V. Shurygin)

[1] *Astrakhan State University, ul. Tatishcheva 20a, Astrakhan, 414056 Russia*
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Abstract—In this paper we describe the structure of the algebra of scalar differential invariants of curves in a plane with Euclidean or Minkowski metric with respect to $\mathbb{R}$-conformal transformations.

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Let $\mathbb{R}^2_\varepsilon$ be a plane with metric $ds^2 = dy^2 + \varepsilon dx^2$. Here x, y are coordinates in the plane and $\varepsilon = \pm 1$. For $\varepsilon = 1$, we have a Euclidean plane, and, for $\varepsilon = -1$, a Minkowski plane.

A transformation ϕ of $\mathbb{R}^2_\varepsilon$ will be called $\mathbb{R}$-*conformal* if it multiplies the metric by a positive constant, i.e., $\phi^*(ds^2) = \lambda ds^2$, $\lambda \in \mathbb{R}^+$.

The set of $\mathbb{R}$-conformal transformations of the plane is a Lie group. We will call this group the $\mathbb{R}$-*conformal* Lie group and denote it by G_{cm}. G_{cm} is a semidirect product of the group of motions G_{m} and the group of homotheties G_{h}.

In the present paper, we give a complete description of the algebra of differential invariants of curves with respect to $\mathbb{R}$-conformal transformations of the plane $\mathbb{R}^2_\varepsilon$.

We introduce the notion of $\mathbb{R}$-conformal curvature of a curve which in the case under consideration plays as important role as the usual curvature in the Euclidean plane. But, in contrast to the curvature of a curve, the $\mathbb{R}$-conformal curvature is a differential invariant of the third order. Differential invariants of kth order are obtained from it by successive applying of the operation of invariant differentiation.

The Lie algebra $\mathcal{G}_{\mathrm{cm}}$ has the basis containing of the following vector fields: $X = \partial_x$, $Y = \partial_y$ (parallel translations), $Z = x\partial_y + \varepsilon y \partial_x$ (rotations[1)]) and $H = x\partial_x + y\partial_y$ (homotheties).

Let φ be a curve in $\mathbb{R}^2_\varepsilon$, given as the graph of a function $y = f(x)$, and let $J^k\mathbb{R}$ be the space of k-jets of smooth functions on $\mathbb{R}$. Recall that a function $I \in C^\infty(J^k\mathbb{R})$ is called a (scalar) *differential invariant* of a curve with respect to a Lie group G if it is not a constant and is preserved under the action of the kth prolongation of the group [1]. The number k is called the *order* of the differential invariant.

Let us find a third order differential invariant of a curve with respect to the group G_{cm}. In order to construct it, we use differential invariants of the group of motions G_{m}. Let x, y, $p_1, p_2, \ldots, p_k$ be the canonical coordinates on the space $J^k\mathbb{R}$. As is known, the first differential invariant of a curve with respect to G_{m} is the curvature of the curve, which is a second order invariant:

$$I_2 = \frac{p_2}{(p_1^2 + \varepsilon)^{\frac{3}{2}}}. \tag{1}$$

*E-mail: strelzova_i@mail.ru.

1)For the Minkowski plane, hyperbolic rotations.