

ON TOPOLOGICAL GROUPS WITH A COUNTABLE SET OF MONOTHETIC SUBGROUPS

Yu.N. Mukhin

Among constraints allowing to select classes of locally compact topological groups G , which can be subject to a study, the constraints in terms of a set (lattice) of all closed subgroups from G , are of a significant interest. Among numerous constraints of that kind, which are already well-studied, one can point to the countability of $L(G)$. In [1], the structure of discrete or compact Abelian as well as inductively compact groups with a countable $L(G)$ was established. The objective of the present article is the study of groups subordinated to a requirement of countability of a certain set of monothetic (i.e., in the topological sense, generated by a single element) closed subgroups. The set of all monothetic subgroups in G (its cardinal number is denoted by $\mu = \mu(G)$, while the cardinality of $L(G)$ by $\lambda = \lambda(G)$) is a basis of the lattice $L(G)$ with respect to the operation of generation of a closed subgroup. As shown in [2] (p. 76), such a basis $M(G)$ in $L(G)$ can be composed by simply taking only \mathbb{Z} -subgroups (which are infinitely discrete cyclic) and cycles ($A \in L(G)$ such that $A(L)$ is a chain with a coatom; these are subgroups either isomorphic to additive groups \mathbb{Z}_p of integer p -adic numbers, or finitely primary cyclic \mathbb{C}_{p^n} , see [3]); the cardinal number $\beta = \beta(G)$ of this set is an invariant (in contrast to μ) of the lattice $L(G)$ (ibid.). Clearly, $\beta \leq \mu \leq \lambda$. A.Yu. Ol'shanskii constructed examples of infinite periodic discrete groups, whose all proper subgroups are simple cyclic, so that $\beta = \mu = \lambda = \alpha$, where α is the cardinal number of a countably infinite set (aleph-null). This motivates the study of groups with at most countable λ , μ , or β under additional assumptions like a generalized commutativity or a generalized finiteness. In the present article to this end we selected the requirement of an inductive nilpotency of the group G . Inductively compact groups with $\beta \leq \alpha$ will be described in another work.

We denote by G_0 a connected component of the unity e of the group G ; by $\Omega(G)$ a set of all finite order elements in G ; $\Pi(G)$ stands for a set of all prime p for which in G a topological p -element exists not equaling e ; G_p stands for a Sylow (Silov) p -subgroup if the former is not unique; $Z_G(M)$ is the centralizer of a subset M in the group G ; \mathbb{T}^m is an m -dimensional torus; \mathbb{Q}_p is the group of all p -adic numbers; $\mathbb{C}_{p^\infty} = \mathbb{Q}_p/\mathbb{Z}_p$ is a quasicyclic group, \times and \times stand for the direct and semidirect topological products, respectively; \simeq means a topological isomorphism. An element a is pure if it generates a subgroup $\langle a \rangle \simeq \mathbb{Z}$.

Examples. 1. Let G be the Tikhonov (Tychonoff) product of finite groups $G_i \neq e$ of mutually prime orders, $i \in I$, $|I| = \alpha$. Here each primary element lies in $\cup_i G_i$, so $\beta(G) = \alpha$. However, $\Omega(G)$, which is the direct product of all G_i , is a subgroup dense in G . Thus, for groups with $\beta \leq \alpha$, one cannot obtain the same structural theorem which holds for groups with $\lambda \leq \alpha$. In the present example, we have the continuum of elements of the form (y_i) , where y_i takes one of the two fixed values g_i or e in G_i , generating various monothetic subgroups, so we have $\mu(G) > \alpha$. On the other hand, if each of all G_i is isomorphic to a single group (for instance, \mathbb{C}_p), then $\beta(G) > \alpha$.

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