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Lower Bounds for Algebraic Algorithms for Nilpotent and Solvable Lie Algebras

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Abstract—We obtain the lower bounds for the tensor rank for the class of nilpotent and solvable Lie algebras (in terms of dimensions of certain quotient algebras). These estimates, in turn, give lower bounds for the complexity of algebraic algorithms for this class of algebras. We adduce examples of attainable estimates for nilpotent Lie algebras of various dimensions.

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1. INTRODUCTION

In this paper we study lower estimates for exact algebraic algorithms. Earlier one considered lower estimates for the class of associative (in particular, matrix) algebras (see, e.g., the survey paper [1]). Estimates for simple Lie algebras are adduced in [2–4].

In this paper we obtain lower estimates for the tensor rank of nilpotent and solvable Lie algebras over fields of zero characteristic, which in turn give lower estimates for the algebraic complexity of algorithms (in this case we mean the algorithms which calculate the product in an algebra) for this class of algebras. See, for example, [5] and similar papers for information about the algebraic complexity and its connection with the tensor rank (the tensor rank does not exceed the complexity). In the case of nilpotent Lie algebras we adduce examples where the obtained estimates are attained.

2. LOWER ESTIMATES FOR NILPOTENT LIE ALGEBRAS

Definition 1. The tensor rank of an algebra is the smallest number r, for which there exist linear functionals $u_i(x_1,\dots,x_n,y_1,\dots,y_n)$, $v_i(x_1,\dots,x_n,y_1,\dots,y_n)$, and elements of the algebra $\overline{w_i}$ such that for any elements $\overline{x}=x_1\overline{e}_1+\dots+x_n\overline{e}_n$ and $\overline{y}=y_1\overline{e}_1+\dots+y_n\overline{e}_n$ the following equality is fulfilled:

$$[\overline{x},\overline{y}]=\sum_{i=1}^{r}u_i(x_1,\dots,x_n,y_1,\dots,y_n)v_i(x_1,\dots,x_n,y_1,\dots,y_n)\overline{w_i},$$

where $\overline{e}_1,\dots,\overline{e}_n$ is some basis of the algebra, considered as a linear space over the field $\mathbb{F}$.

We denote the tensor rank of an algebra L by $\mathbf{rk}_{\otimes}(L)$, we do the center of the algebra by $\mathbf{Z}(L)$, and we do the main field (of zero characteristic) by $\mathbb{F}$. We write the tensor of the product of an arbitrary algebra L in the form

$$[a,b]=\sum_{i=1}^{r}u_i(a,b)v_i(a,b)w_i, \qquad (1)$$

where r is the minimal of possible ranks (here $r=\mathbf{rk}_{\otimes}(L)$); $u_i(a,b)$, $v_i(a,b)$ are linear functionals on $\langle L\times L\rangle$; $a,b,w_i\in L$.

Consider an arbitrary nilpotent Lie algebra L over the field $\mathbb{F}$ of zero characteristic.

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