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HORIZONTAL LIFTS OF TENSOR FIELDS TO SECTIONS OF TENSOR BUNDLE

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1. Introduction

Let M_n be an n-dimensional differentiable manifold of class C^∞, and $T^p_q(M_n)$, $p+q>0$, the bundle of tensors of type (p,q) on M_n. The aim of this article is to study horizontal lifts of tensor fields from M_n to the tensor bundle $T^p_q(M_n)$ along sections of this bundle. We apply a new method with the use of the Vishnevskiĭ operator.

We use the following notation.

1. $\pi : T^p_q(M_n) \to M_n$ is the projection of $T^p_q(M_n)$ onto M_n.

2. The index ranges are as follows: i, j, k run through $1,\dots,n$; $\bar{i}$, $\bar{j}$, $\bar{k}$ run through $n+1,\dots,n+n^{p+q}$; $I=(i,\bar{i})$, $J=(j,\bar{j})$, $K=(k,\bar{k})$ run through $1,\dots,n+n^{p+q}$.

3. $\mathcal{F}(M_n)$ is the ring of smooth real-valued functions on M_n. $\mathcal{T}^p_q(M_n)$ stands for an infinite-dimensional vector space over $\mathbb{R}$ of smooth tensor fields of type (p,q). We also will consider $\mathcal{T}^p_q(M_n)$ as a module over the ring $\mathcal{F}(M_n)$.

4. We denote by $V, W, \dots$ vector fields on M_n, and by φ a tensor field of type $(1,1)$.

2. Horizontal lifts of vector fields to sections

Let us denote by x^j local coordinates in a neighborhood $U \subset M_n$, and assume that $x^{\bar{j}} \overset{\text{def}}{=} t^{i_1\dots i_p}_{j_1\dots j_q}$ are the induced coordinates on $\pi^{-1}(U) \subset T^p_q(M_n)$.

A tensor field $\alpha \in \mathcal{T}^q_p(M_n)$ determines with the use of contraction a function on $T^p_q(M_n)$, call it $i\alpha$. With respect to the local coordinates, if $\alpha = \alpha^{j_1\dots j_q}_{i_1\dots i_p}\partial_{j_1}\otimes\dots\otimes\partial_{j_q}\otimes dx^{i_1}\otimes\dots\otimes dx^{i_p}$, then $i\alpha(t) = \alpha^{j_1\dots j_q}_{i_1\dots i_p} t^{i_1\dots i_p}_{j_1\dots j_q}$ for each $t \in \mathcal{T}^p_q(M_n)$.

Let $A \in \mathcal{T}^p_q(M_n)$. We define the vertical lift ${}^VA \in \mathcal{T}^1_0(T^p_q(M_n))$ of A to $T^p_q(M_n)$ (see [1]) by the requirement ${}^VA(i\alpha) = \alpha(A)\circ\pi = {}^V(\alpha(A))$, where ${}^V(\alpha(A))$ is the vertical lift of the function $\alpha(A) \in \mathcal{F}(M_n)$. With respect to $(x^j, x^{\bar{j}})$ the vertical lift ${}^VA$ of A to $T^p_q(M_n)$ has the coordinates

$$ {}^VA = \begin{pmatrix} {}^VA^j \\ {}^VA^{\bar{j}} \end{pmatrix} = \begin{pmatrix} 0 \\ A^{i_1\dots i_p}_{j_1\dots j_q} \end{pmatrix}. $$

Suppose that on M_n a torsion-free affine connection ∇ is given. Let ∇_V be the covariant derivative with respect to $V \in \mathcal{T}^1_0(M_n)$. The horizontal lift ${}^HV = \overline{\nabla}_X$ of a vector field V to $T^p_q(M_n)$ (see [1]) can be defined by the requirement ${}^HV(i\alpha) = i(\nabla_V\alpha)$, $\alpha \in \mathcal{T}^q_p(M_n)$. With respect to the coordinates $(x^k, x^{\bar{k}})$ the components of ${}^HV$ can be written as follows:

$$ {}^HV^k = V^k, \quad {}^HV^{\bar{k}} = V^m\left(\sum_{\mu=1}^{q}\Gamma^s_{mk_\mu} t^{l_1\dots l_p}_{k_1\dots s\dots k_q} - \sum_{\lambda=1}^{p}\Gamma^{l_\lambda}_{ms} t^{l_1\dots s\dots l_p}_{k_1\dots k_q}\right), $$