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Inverse Boundary-Value Problems of Cauchy Type for Harmonic Functions

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Abstract—We apply two methods for solving the inverse boundary-value problem (the so-called problem (A)) in the Cauchy statement for an analytic function and an unknown curve Γ. We obtain criteria for Γ to be the unit circle. We apply the proposed methods for solving a modified Hadamard example and generalize the obtained results for the case of doubly connected domains.

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A. S. Demidov and D. A. Platushchikhin [1] studied the Cauchy problem for the two-dimensional Laplace equation and obtained the function

$$z(\zeta) = P_0 + \int_1^{\zeta} \exp(A + iB)\, d\zeta, \tag{1}$$

which is analytic in the annular neighborhood $E(r_1, r_2) = \{\zeta : r_1 < |\zeta| < r_2\}$, $r_1 < 1 < r_2$, of the unit circle $|\zeta| = 1$ and maps this circle on a curve Γ. The conditions of the Cauchy problem are presented in the form

$$u|_\Gamma = f(s) = \operatorname{Re} \sum_{k\geq 0} f_k e^{iks}, \quad \frac{\partial u}{\partial \nu}\Big|_\Gamma = g(s) = \operatorname{Re} \sum_{k\geq 1} g_k e^{iks}, \tag{2}$$

where s is the length of the arc (the arc coordinate) of the curve Γ, $0 \leq s \leq 2\pi$, ν is the outer normal at points of the curve Γ bounding a finite simply connected domain Ω. The curve Γ is determined in terms of the function $Q(s) = N(s) - s$, where $N(s)$ is the angle between the abscissa axis and the normal ν at the point $P_s \in \Gamma$; one can also use a natural equation of the curve Γ with $|dz/d\zeta| = 1$.

Problem (A) implies the determination of the curve Γ and the analytical function $z(\zeta)$ in the form (1) from boundary data (2). It has a unique solution. We establish properties of functions $f(s)$ and $g(s)$ that turn Γ into the unit circle (Theorem 1), describe the process of solving problem (A) by the Demchenko scheme, propose a new approach to its solution (Theorem 2), and reduce problem (A) to the basic inverse boundary-value problem by means of the Tumashev–Nuzhin method (Theorem 3). We also solve a modified Hadamard example in various ways and present preliminary results on the generalization of problem (A) for the case of a doubly connected domain.

Note that the solution of the Cauchy problem after the construction of the curve Γ is $u = \operatorname{Re} \widetilde{f}[\zeta(z)]$; see below its special form (10) for the Hadamard example.

1. The uniqueness of a solution to problem (A). We say that two curves Γ and Γ_1 give one and the same solution to problem (A), if Γ_1 can be obtained from Γ by the rotation and translation. One can easily prove the following lemma.

Lemma. *The solution to problem* (A) *is unique.*

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