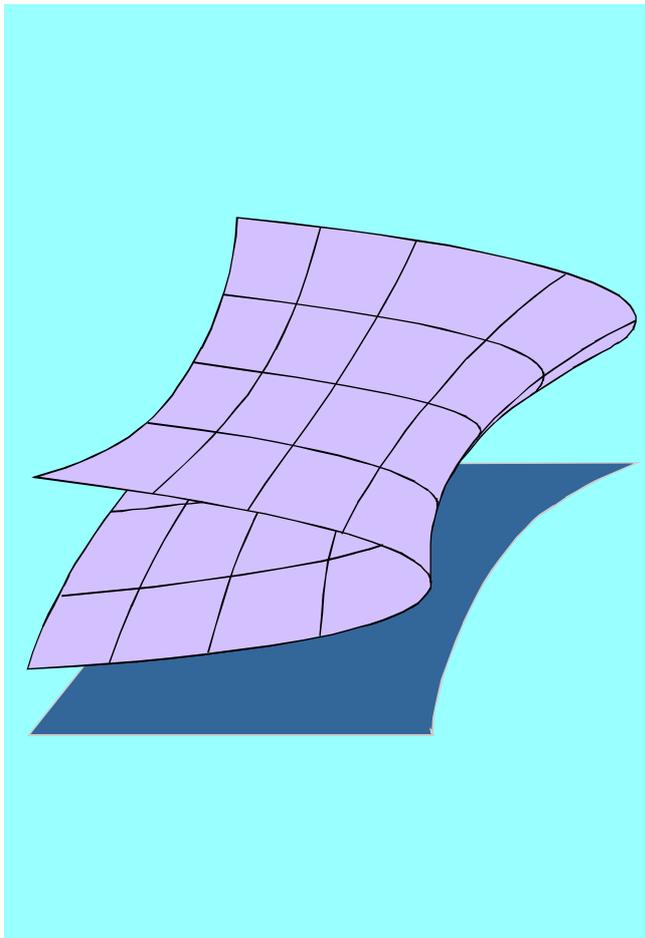


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- to promote close working contacts between scientists of various Universities and Schools; between theorists and application oriented scientists;
- to mathematize the methods for solving the problems generated by engineering practice;
- to unite the efforts, to synthesize the methods in different areas of science and education.

In Journal the articles and reviews; the engineering notes; the discussional communications; the statements and solutions of problems in all areas of nonlinear analysis and their applications in engineering systems are published (including new results, methods, approaches, hypotheses,...). Authors of theoretical works should indicate the possible areas of applications in engineering practice.

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### **From Editorial Board**

International Journal “Problems of Nonlinear Analysis in Engineering Systems” is an *interdisciplinary bilingual* scientific periodical Edition, which represents the research of nonlinear problems in general, in the whole diversity of fundamental and applied sciences, including the disciplines of natural science and the Humanities.

Note, close *interdisciplinary* relation between the fundamental and applied fields of science, between different disciplines has vital importance for the development of our Knowledge in whole. *Fundamental science* “MECHANICS” and its sections play an important role from this point.

Prominent role of Mechanics as a fundamental basic scientific discipline for all another disciplines and for our Knowledge in whole is unquestionable. Science “Mechanics” is investigating the motion and interactions of objects; and “Mechanics” supplies us with models and methods that are covering all areas of theory and engineering:

*“Newtonian mechanics is an unequalled achievement of physics (natural philosophy), the whole history of human civilization. It is everlasting. Its powerful tree is sprouting more and more branches. Among them there are the branches that have grown from scions grafted on this tree and cultivated in other natural sciences”, - G.G.Chyorny, Academician of Russian Academy of Science (Chairman of Russian National Committee on theoretical and applied mechanics, 2011).*

- Mechanics has been the main tool for the research of celestial bodies and all the processes in near-Earth and outer Space.
- “Mechanics” provides powerful tools for description of processes at micro level; also – for description of complex processes inside the Earth, in geodynamics; in the study of volcanic eruption processes, hurricane dynamics; in Arctic exploration,...
- Methods and models of Mechanics are giving very effective approaches for multidisciplinary engineering applications, including various domains: medicine, robot-technique, biomechanics, mechanics of materials, nano-biotechnology, complex problems of designing in aviation and space technique, ...
- The Mechanics is «the main foundation» to development of all adjacent disciplines, in which studied objects are *the interdisciplinary* systems requiring knowledge from various scientific areas. Exactly on boundary between different disciplines the new hypotheses are generated, that providing deep knowledge of World around, with understanding of occurring phenomenas.
- Without Mechanics, without close interdisciplinary relations between theoretical and applied areas, between different disciplines of the Science, the deepening our Knowledge in whole is impossible.

Moreover exactly Mechanics is promoting the development of “*mathematical constructions of exclusive beauty*”: the dynamic systems theory, A.M.Lyapunov stability theory (which 120-th Anniversary was celebrated in 2012).... And it plays in all this the major role, with uniting efforts of theorists and engineers, with development and synthesis of methods for the solving problems in interdisciplinary spheres of a science, education and engineering practice, in the research of nonlinear problems in general, in the whole diversity of fundamental and applied sciences including the disciplines of natural science and the Humanities (mathematics, mechanics, physics, chemistry; engineering, biological, medical, social, political sciences; ecology, cosmology; economics and financial mathematics; nanoscience

and nanotechnology, stability and sustaining development, problems of risk and information security, operation research, ...).

*“... The stability theory and dynamic properties analysis of nonlinear systems – it is magnificent tree, possessing the classical stem, the deep strong roots from Mechanics, ..., from important engineering problems,...”*, - V.M.Matrosov, Academician of Russian Academy of Science (President of Academy of nonlinear sciences, 2001).

In this direction it is very important the A.M.Lyapunov-N.G.Chetayev methodology, developed for problems of modelling and analysis in engineering practice and for extending our Knowledge in whole. The A.M.Lyapunov stability theory is giving for us the constructive mathematical tool, and it is confirming:

“mathematics is an effective “transport” which is able to provide significant breakthrough in understanding of the essence of Environment, with deep penetration of its approaches into all the spheres including the unconventional ones”.

*“...I always believed that the objective character of Self Organization and Irreversibility should be based on some qualitative properties of **Dynamics**; ...Universe is a construction in progress, in which **we participate**”*, - I.Prigogine, Nobel Laureate.

The current issue of “Problems of Nonlinear Analysis in Engineering Systems” (№1 (39), т.19, 2013) carries articles, analytical researchers and authors results, scientific and information papers that reflect the views of specialists and highlight some topical *interdisciplinary* problems of present and future.

Among them it is presented the papers and reviews of interdisciplinary subjects, scientific research and interdisciplinary spheres generated by the needs of fundamental science and engineering applications. These articles are prepared in the development of the researches results discussed at the International scientific forums and conferences, including World Congress of Nonlinear Analysts (WCNA-2012), within invited scientific Session “Problems and methods of modelling and analysis in Complex multidisciplinary Systems Dynamics”, devoted to 120-th Anniversary of “Stability theory of A.M.Lyapunov” and to Memory of great, brilliant Scientists – N.G.Chetayev (110 years); I.R.Prigogine, NOBEL LAUREATE (95 years); V.M.Matrosov(80 years); XXIII International Scientific Workshop (2013) on problems of modelling and dynamics of complex multidisciplinary systems.

The subjects represented in these works: methods and models of interaction of galactic comets with the Earth, models in studying the physical phenomena of heating lithospheric rocks by galactic comets; continuum mechanics, with discussion of a correctness of classical statements and models; development of analytical solutions of Korteweg – de Vries equations, with using time-fractional models; methods for study of properties and applications of curvature, conformal metrics; dynamics problems of solids systems with friction,...

The submitted articles will undoubtedly promote the cooperation of specialists in theory and applications, support the synthesis of approaches to the solution of problems in interdisciplinary spheres of science, education and engineering practice.

The issue is prepared with support of our Partners: International Federation of Nonlinear Analysts, Academy of Nonlinear Sciences, International Nano-biotechnology Center (INT), Kazan Federal University (KFU), N.E.Bauman Moscow State Technical University, Moscow Aviation Institute (National Research University), V.F.Trapeznikov Institute of control problems of RAS, A.A.Dorodnitsyn Computing Centre of RAS; Concern CSRI Elektropribor, JSC; TsNIImash.

# **Study of the mechanism heating lithospheric rocks by galactic comets using Lavrentiev's hydrodynamics model**

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We study the mechanism of heating lithospheric rocks by shock waves from galactic comets, destroyed in the Earth's atmosphere. For the theoretical calculation effects of heating rocks are used hydrodynamic model of ideal fluid, developed to study collision phenomena at cosmic speeds by M.A.Lavrentiev. We obtained estimates size of crater as well as depth and volume of igneous chambers that are created by galactic comets with different energy. We also are discussing main tectonic-igneous processes in the Earth's crust and mantle, initiated by galactic comets falls [1-24].

## **1. Introduction**

Galactic comets are the class of large space bodies that bombard Earth and other planets in the periods when Solar system stands in jet flows of the Galaxy [1, 2]. Duration bombardment of  $\sim 1 \div 5$  million years, and they are repeated every 20-37 million years. During each bombing on Earth may fall  $\sim 0^4 \div 10^7$  comets galactic origin. These comets are moving relative to Sun at speed of 450 km/s, composed primarily of water ice density of  $1 \text{ g/cm}^3$ , and the diameter of their nucleus varies from 100 to 3500 m. The consequences fallings of such comets onto the celestial bodies with no atmosphere (Moon, Mercury), and also Mars, which has a much diluted gas shell, and on the planet with dense atmosphere (Earth and Venus) are different. If in first case, comets form craters with diameters of 10-200 km, then in the second they lift surface of planets on large areas and are generating seamounts (on Earth) and shield volcanoes (on Venus).

The great efforts have been made in recent years to explaining why the falls of galactic comets on planets with an atmosphere and without are so strongly differ. At now known that galactic comets are destroyed in Earth's atmosphere and reach the surface our planet in the form of hypersonic gas jet consisting of vaporized material of comet and shock heated air [4]. This gas jet is almost no loss of initial velocity reaches planet's surface, but does not create conspicuous crater. Thus, physical mechanism interaction of galactic comets with Earth are fundamentally differs from asteroids and comets, which, when falling to Earth surface at a speed of 20-70 km/s, cause formation of large craters [3-6].

Geological data suggest that galactic comets capable create in lithosphere's rocks the narrowcasting hypersonic shock wave which incurs most of the comet's energy. This shock wave penetrates enough deep in lithosphere and gives its energy to environment, causing heating and partial melting of rocks. This hypothesis can explain a number of important tectonic-igneous processes on Earth [7] and on Moon [8], which had been initiated by last cometary bombardment during period at 5 to 1 million years ago, as well as of earlier periods of geological history [2].

However the physical mechanism of heating rocks by cometary shock waves theoretically studied still insufficient. To solve this problem the author used hydrodynamic model developed by M.A.Lavrentiev for theoretical description of collision processes in case cosmic velocities [9]. This model is based on the hypothesis that collision of bodies moving at cosmic speeds can be considered as an interaction of two ideal incompressible fluids.

In collisions at very high speeds, produced pressure is so great, what we can take no into account strength and plastic properties of the environment and forces of friction compared to

inertial forces. If we also will defy compressibility of the environment, then we will get the model of ideal incompressible fluid [10].

We assume that given Lavrentiev's theory also allows us to progress in the understanding series processes that are initiated by the falls galactic comets to Earth. In this paper we are substantiated the possibility of attraction this model to study the mechanism interaction galactic comets with Earth's lithosphere. We have calculated thermal effects from falls to Earth galactic comets different masses, and shown that, based on developed approach we can from a new angle to approach to solution some key geophysical and geochemical problems.

## 2. The theoretical model of Lavrentiev

We present formulation and solution this problem by Lavrentiev [9] in one-dimensional case when calculation scheme is particularly simple. Let us consider a plate thickness  $a$ , flying at speed of  $V_0$ , which strikes in end surface rod of a length  $L$  (where  $L \gg a$ ). The impact is inelastic. Both bodies are considered to be absolutely rigid and incompressible.

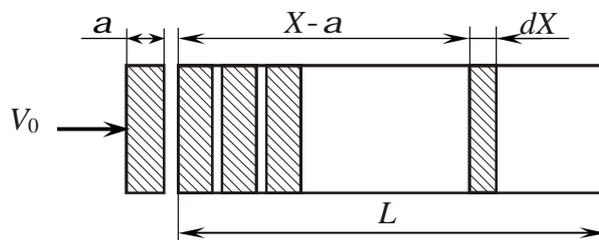


Fig.1. Statement of problem by Lavrentiev [9].  
 $a$  – thickness of plate,  $V_0$  – its speed,  $L$  – length of rod,  $X$  – current coordinate.

The rod presents in form stack of plates, when their thickness tends to zero, and the number grows without bound, so that total amount is equal to  $L$  (fig.1). For simplicity, the matter density of plate  $a$  and rod assumed to be 1. The task is to find kinetic energy loss along rod, as well as momentum, which rod gets in result of impact.

Let at given moment of time piece of rod by length  $X$  takes part in movement with speed  $V$ . When, next segment of rod  $dX$  will be involved in movement, this speed changes on value  $dV$ . On the basis of the law of conservation of momentum we can write

$$X \cdot dV + V \cdot dX = 0 \tag{1}$$

Noting that at initial condition  $V|_{X=a} = V_0$ , distribution of speed along the rod after integration (1) we will obtain in form

$$V(X) = V_0 \frac{a}{X} \tag{2}$$

In according with depending (2) the distribution energy in rod is described by formula

$$E(X) = \frac{XV^2}{2} = \frac{V_0^2}{2X} = \frac{E_0}{X} \tag{3}$$

where  $E_0 = aV_0^2/2$  –initial energy of plate-peen

Assuming that all lost energy is converted into heat, density distribution of heat along rod can be calculated by formula

$$U(X) = -dE/dX = aE_0/X^2 \tag{4}$$

Let denoted by  $U_{cr}$  minimum density of heat, at which substance of rod goes into gas. Then process of energy transfer will take place only as long as the  $X$  reaches value

$$X_{cr} = \sqrt{aE_0/U_{cr}} \quad (5)$$

In this point the part of rod length  $0 < X < X_{cr}$  turn into gas. This gas at extension is separating from the rest of rod. For the most interesting case, when  $a \ll X_{cr}$  and  $X_{cr} \ll L$ , the amount of energy expended to transform a piece of rod in the gas is small compared to the initial energy  $E_0$ . Therefore we can assume that all the energy of gas goes into its kinetic energy. In this case, speed of expansion of a gas cloud  $W$ , obtained from equation  $X_{cr} W^2 = aV_0^2$ , will be equal

$$W = \sqrt{aV_0/X_{cr}} \quad (6)$$

The rest of the rod retains part of energy wave and gains some momentum. Lavrentiev approximately calculates magnitude of this momentum for two extreme cases: a) in assumption that all particles of the gas cloud have equal speed:

$$I = X_{cr} \cdot W = \left( \frac{E_0}{U_{cr}} \right)^{1/4} a^{3/4} V_0 \quad (7)$$

and b) when each layer of gas is emitted in  $X$  direction independently. Then momentum layer at a distance  $X$  from end of cylinder is equal  $VdX = \sqrt{2U_{cr}}dX = \sqrt{2aE_0} \frac{dX}{X}$  and total momentum will

$$I = \sqrt{2aE_0} \log \frac{X_{cr}}{a} = \sqrt{2}aV_0 \log \frac{V_0}{\sqrt{U_{cr}}} \quad (8)$$

Let us now apply solution this task to assess effects of fall to Earth galactic comets.

### 3. Fall galactic comet to Earth

The complex process interaction comets with Earth place we partition in two phases: the stage aerodynamic destruction of comet's nucleus in Earth atmosphere and the stage propagation cometary shock wave in rocks of lithosphere. The solution of task on first phase will be considered as initial conditions of problem second stage.

#### 3.1. Stage comet destruction in atmosphere

As a solution problem of first stage, we use results of our work [11] in which performed physical-mathematic modeling destruction galactic comet passing through aerial shell of our planet (fig.2).

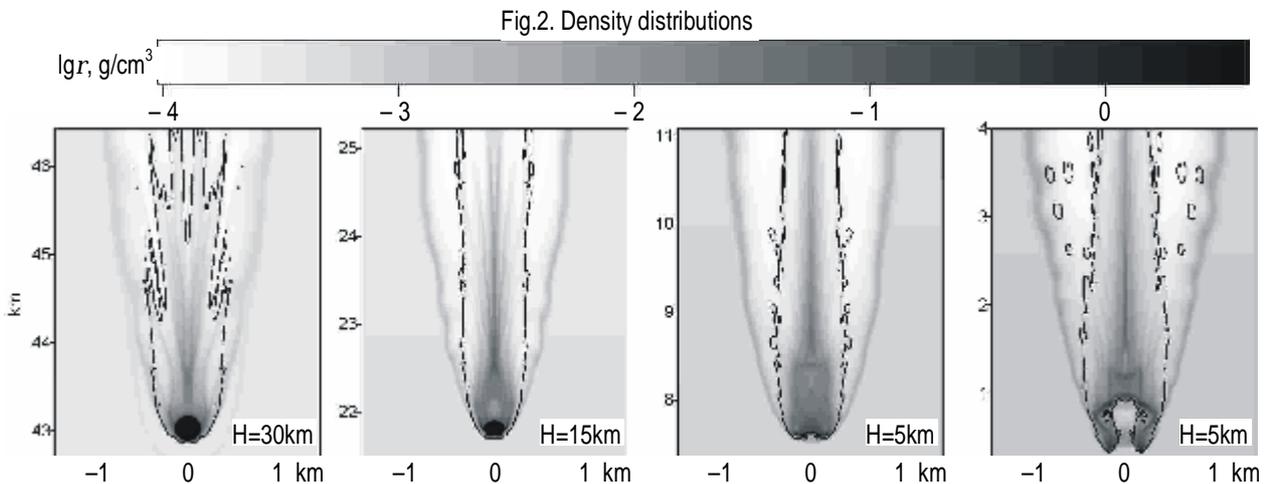


Fig.2. Temperature distributions

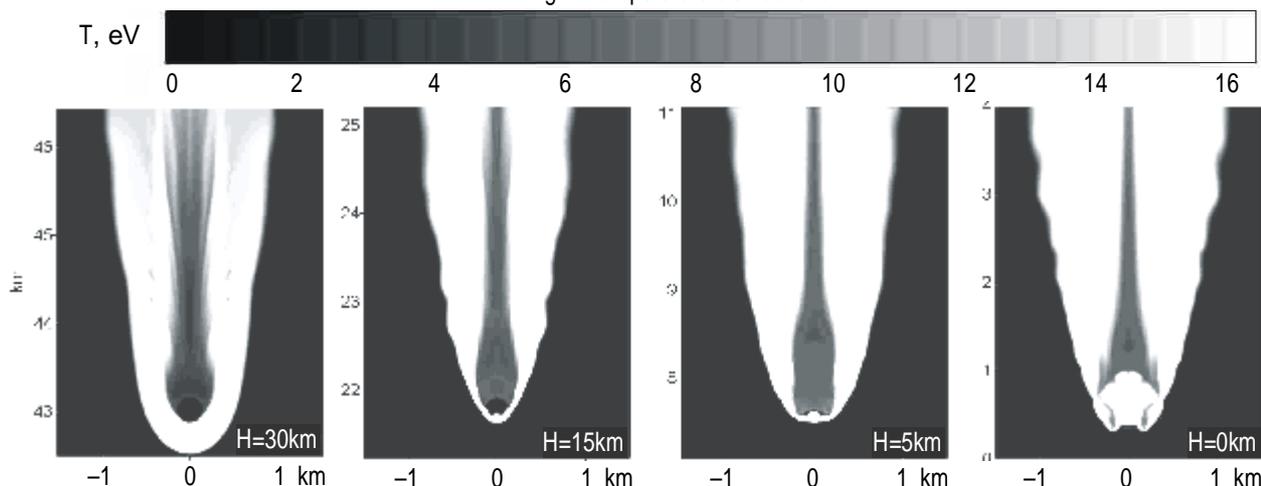


Fig. 2. Disintegration in Earth's atmosphere galactic comet of diameter 300 m, which falls to terrestrial surface at angle  $45^\circ$ . Shown distributions density and temperature at different heights  $H$ . On the vertical axis is indicated the distance along trajectory. By thickened line is shown the boundary between comet's vapors and air.

We modeled falling icy comet diameter of 300 meters, flying at speed of 450 km/s at  $45^\circ$  to Earth's surface. The comet nucleus was taken quasi-liquid (with zero durability). Deformation of the comet nucleus under action of aerodynamic load was described by Euler's hydrodynamic equations. Radiation transport in comet's pairs and in air was calculated in the approximation of radiant heat-conductivity. The pressure of cometary vapors was equaled to pressure of gas near cometary nucleus surface. Temperature and density of vapor were determined from the phase equilibrium curve on base of specially calculated tables of state equations for both substances in a wide range of densities and temperatures. Numerical integration of the Euler equations was performed using program [12]. She allowed integrate equations of gas dynamics in Eulerian form by means two step method on an irregular movable grid in Cartesian and cylindrical coordinates, inclusive of boundaries between media with different equations of state.

Two-dimensional problem was solved with the axis  $z$ , directed along comet's trajectory. The slope of trajectory was taken into account by corresponding increase of characteristic height of atmosphere. The computational grid consisted of  $200 \times 600$  cells in the  $r$  and  $z$  directions. Mesh density in the central region was  $70 \times 100$  cells. As the distance from  $z$ -axis resolution of grid has increased exponentially. Grid moved along  $z$ -axis with comet.

Modeling lets to trace the main stages of transformation of comet's nucleus in atmosphere (fig.2). According our calculations, at a height of over 50 km around comet's nucleus formed a dense cloud of vapors, which fills comet trail. Before the bow shock, which in these conditions is supercritical, is formed heated layer, whose thickness is several times the size of comet's nucleus. At altitudes below 30 km, due to emergence of large aerodynamic loads on frontal surface of comet nucleus, exceeding limit of mechanical strength of ice, comet's nucleus begins to deformed and disintegrate. Due to development instabilities of Rayleigh-Taylor and Kelvin-Helmholtz at surface nucleus appear wavelike disturbances, and it starts to rapidly lose mass. In result destruction of periphery areas of comet's nucleus and their inhibition in air, a significant mass of comet's substance is lost. This loss of mass at ablation grows larger than loss of material at evaporation comet's ice.

The growth of aerodynamic pressure leads to rapid fragmentation and flattening of comet's nucleus due to which the nucleus on height  $\sim 5-7$  km turns into crushed pancake structure. Further development of instabilities causes to fully disintegration cometary nucleus. At

heights lower ~3-5 km nucleus transforms into a jet consisting of vapor, drop-liquid and small solid particles as well as air, heated by front shock wave. At a height of ~1 km the stream transforms into pure gas jet which strikes upon Earth's surface. At this jet velocity differs from initial comet velocity no more than on 10%.

Of greatest interest to us are parameters of frontal part jet on final trajectory. We see that, starting from distance of ~5 km from terrestrial surface, gas stream in the first approximation is a cylinder, which diameter becomes about twice more then diameter of comet's nucleus. At the very surface in front part of gas jet appears recess.

### 3.2. Stage interaction cometary jet with lithosphere

In contrast to simplified formulation of task by Lavrentiev [9], in the second stage of our calculations we take into account length cometary jet, as well as different substance density of jet and terrestrial rocks. In determining parameters comet's jet we use the results simulations of passing comet of diameter  $d_k = 300$  m through Earth atmosphere. On basis of data (fig.2), we assume that comet is completely transformed into a cylindrical gas jet (recess disregarded), which falls normally on Earth surface. Thus, to a first approximation we can assume that substance density of the jet  $r_0 = m_k / (p d_k^2 a)$ . The dense part of jet has length  $a = 1$  km and its diameter is approximately 2 times more diameter cometary nucleus. This leads to impact area  $S = p d_k^2 = 2.8 \cdot 10^9$  cm<sup>2</sup>. We also assume that on Earth surface (although with some delay) drops almost all mass cometary material. The jet speed and its kinetic energy remain same as for galactic comet and respectively equal  $V_0 = 450$  km/s and  $E_0 = 1.4 \cdot 10^{21}$  J. As in Lavrentiev's task, we will considered blow gas jet in Earth surface is inelastic, so all energy comet  $E_0$  transfers into wave moving along a cylindrical block of rock. For the convenience of our next calculations thickness plates (fig.1) we equate 1 cm.

As material of lithosphere we shall take a gabbroid anorthosite. Its density in solid state we shall accept  $r_{sol} = 3.3$  g/cm<sup>3</sup>, and in molten  $r_{melt} = 2.7$  g/cm<sup>3</sup>. The specific heat evaporation of anorthosite we shall accept  $q_{vap} = 5.7 \cdot 10^4$  J/g and its heat of fusion  $I_{melt} = 3.4 \cdot 10^3$  J/g [4]. And at last we supplement this data the specific coefficient of heating rocks  $C = 1.2$  J/K, as well as their coefficient of volume expansion  $b = 10^{-5}$  (K)<sup>-1</sup>.

As and in case fallings of ordinary asteroids and comets [4], we distinguish two main stages of shock process: phase propagation shock wave in lithosphere and phase modification column heated rocks.

**3.2.1. Phase propagation shock wave.** According to Lavrentiev's theory in result passage shock wave are formed three zones:

- I – evaporation zone rocks,
- II – zone complete melting rocks, and
- III – zone of heating rocks.

We will calculate size of each zone and shall estimate the basic parameters of substance of the last two zones. For that we use formulas (1-6). Since our task is two-dimensional and in addition densities of gas jet and lithospheric rocks differ, we introduce in these formulas additional multipliers.

**Zone I (crater).** When calculating length column of evaporated rocks, we will define the value  $U_{cr}$  in (5) as quantity of energy required for evaporation layer rocks of 1 cm. The mass this layer rocks is  $m_1 = r_{sol} \cdot S \cdot 1$  cm =  $9.24 \cdot 10^9$  g, and the energy required for its evaporation is  $U_{vap} = q_{vap} \cdot m_1 = 5.3 \cdot 10^{14}$  J/cm.

Substituting the value  $U_{vap}$  in (5), length column of evaporated rocks taking into account different density of substance gas jet and anorthosite will defined like

$$X_{vap} = (a r_0 E_0 / r_{sol} U_{vap})^{1/2} = 0.63 \text{ km.}$$

This value determines depth of crater. Mass evaporating rocks will be  $m_{\text{vap}}=X_{\text{vap}}\cdot m_1=5.8\cdot 10^{14}$  g, which is almost 50 times large mass of galactic comet. Gaseous material will be thrown from cylindrical crater with speed

$$W = V_0 (ar_0/X_{\text{vap}}\cdot r_{\text{sol}})^{1/2} = 70 \text{ km/s.}$$

On that will be expended energy

$$E_{\text{vap}} = E_0 - E(X_{\text{vap}}) = E_0 \cdot [1 - (ar_0/X_{\text{vap}}\cdot r_{\text{sol}})] = 1.37 \cdot 10^{21} \text{ J.}$$

The remaining comet's energy will pass in underlying rocks, which will heat up and melt.

**Zone II (igneous chamber).** The lower boundary of the zone melting of rocks also may be calculated by the refined formula (5). But now under value  $U_{\text{cr}}$  we will understand the heat of fusion anorthosite which determined like  $U_{\text{melt}}=I_{\text{melt}}\cdot m_1=3.14\cdot 10^{13}$  J/cm. The result will be:

$$X_{\text{melt}} = (ar_0E_0/r_{\text{sol}}U_{\text{melt}})^{1/2} = 2.6 \cdot \text{km.}$$

The power layer molten rocks, defining length igneous chamber, will be

$$\Delta X_{\text{melt}} = X_{\text{melt}} - X_{\text{vap}} = 1.97 \text{ km.}$$

Quantity melted rocks melt rocks may be calculated like  $m_{\text{melt}} = m_1 \cdot \Delta X_{\text{melt}} = 1.82 \cdot 10^{15}$  g, and energy spent on creation igneous chamber defined like  $E_{\text{melt}} = E(X_{\text{vap}}) - E(X_{\text{melt}}) = 2.54 \cdot 10^{19}$  J. Part this energy  $Q_{\text{melt}}=I_{\text{melt}}\cdot m_{\text{melt}} = 6.2 \cdot 10^{18}$  J will go on melting rocks and the rest  $\Delta E_{\text{melt}} = E_{\text{melt}} - Q_{\text{melt}} = 1.92 \cdot 10^{19}$  J causes heating melt. Note that the energy needed to heat molten rocks more than 3 times higher than energy their melting.

**Zone III.** The rest comet's energy equal to  $E(X_{\text{melt}}) = 8.16 \cdot 10^{18}$  J will be used for heating column rocks deeper than 2.6 km. Density distribution this energy along column rocks also may be described by means refining formula (4). However we primarily would like to know increasing temperature and volume rocks at heating.

According to (4) temperature heating of rocks varies with depth for  $X \geq 2.6$  km like

$$T(X) = T_{\text{melt}} \cdot (X_{\text{melt}}/X)^2 \quad (9)$$

where:  $T_{\text{melt}} = I_{\text{melt}}/C = 2.83 \cdot 10^3$  K – temperature complete melting rocks.

If we restrict heating zone rocks for example up 100 K then according to (9) boundary this zone will be located at depth  $X_{100}=X_{\text{melt}}(T_{\text{melt}}/100)^{1/2}=13.8$  km. Height column heated rocks will then reach  $\Delta X_{\text{(III)}}=X_{100}-X_{\text{melt}}=11.2$  km, and their volume will be equal  $G_{\text{(III)}}=S\cdot\Delta X_{\text{(III)}}=3.14$  km<sup>3</sup>.

The results calculations are shown in fig. 3. For ease analysis data results of calculations are given in coordinate plane  $\{T, X\}$ , where the ordinate is presented temperature rocks in degrees Celsius, and on the horizontal axis – depth rocks occurrence. Area shaded triangles characterize amount thermal energy transferred by shock wave into zones II and III.

Velocity shock wave decreases with depth. In particular, on border zones II and III this speed, by formula (2), is equal  $V(X_{\text{melt}}) = (a/X_{\text{melt}}) \cdot V_0 = 173$  km/s, and at depth  $X_{100}$  decreases to value  $V(X_{100}) = 32.6$  km/s. Both these values are multiply higher then speed longitudinal seismic waves at these depths at 6÷8 km/s.

Time movement shock wave through all length column  $\Delta X_{\text{(III)}}$  can be approximately estimated like

$$t \approx 2\Delta X_{\text{(III)}}/[V(X_{\text{melt}})+V(X_{100})] = 0.11c \quad (10)$$

This time is estimate duration entire phase heating rocks by shock wave.

**3.2.2. Phase modification.** At this phase, the thermal energy stored in column rocks after passage shock wave is redistributed in environment. This thermal energy causes volumetric expansion rocks, which shifts boundaries of three zones, and in finally leads to change crater's depth. Duration phase modification is many longer travel-time of shock wave.

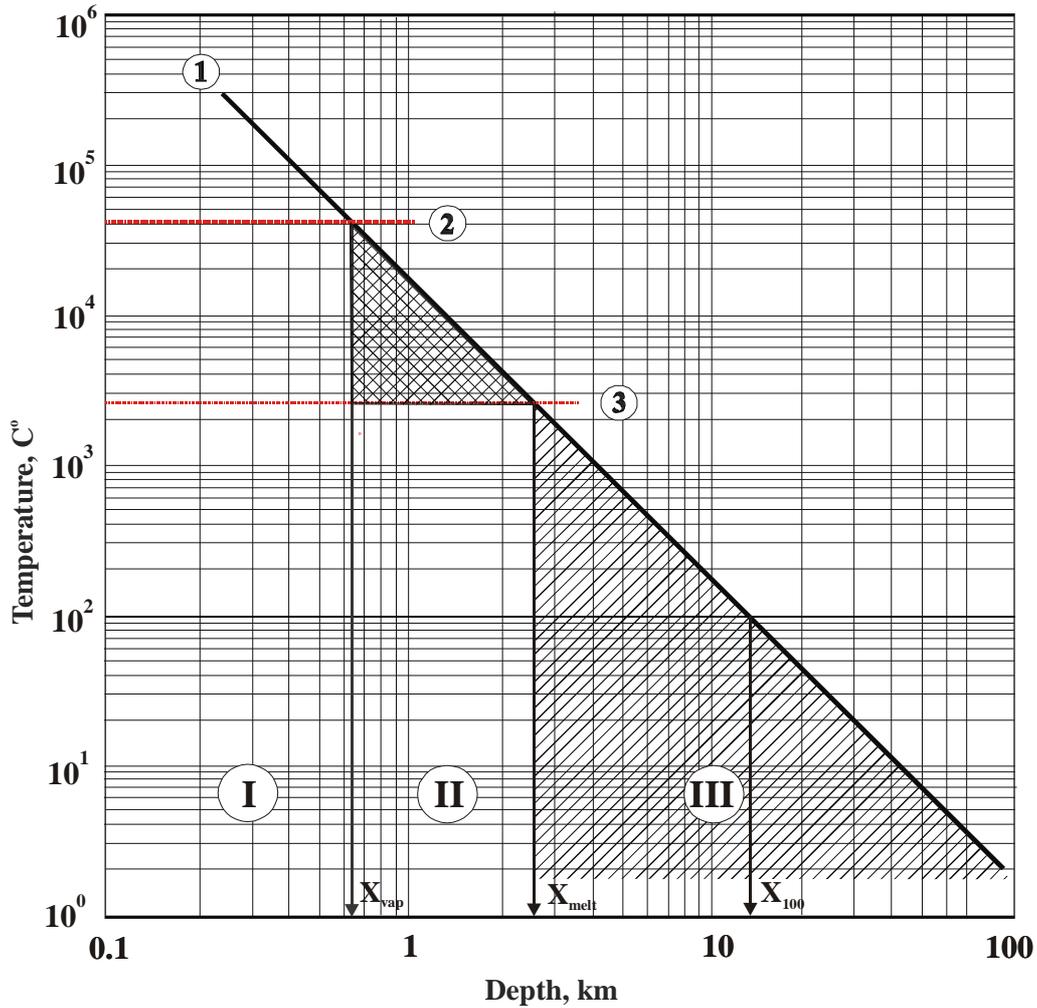


Fig.3. Heating rocks by shock wave created by comet with diameter of 300 m. Legend: 1 – calculated temperature dependence, 2 – evaporation temperature of rocks, 3 – melting point of rocks. Area of shaded triangle characterizes energy of heating rocks. Roman numerals indicate the zone of rocks: I – evaporation, II – melting and III – heating up to 100°C;  $X_{vap}$ ,  $X_{melt}$  and  $X_{100}$  – borders of zones.

Volumetric expansion of rocks is most easily calculated for zone III. The change in volume rocks this zone can be calculated like:

$$DG_{(III)} = b \cdot S \int_{X_{melt}}^{\infty} T(x) dx = b \cdot T_{melt} \cdot S \cdot X_{melt} = 2.1 \cdot 10^{12} \text{ cm}^3 \quad (11)$$

Since rocks can expand only in direction of free surface, their heating in our model will lead to offset lower boundary of molten rock at  $dX_{heat} = DG_{(III)}/S = 75$  m. Upper boundary melting zone rocks  $X_{vap}$  that defines crater's depth, experiences even more displacement. Since below this border rocks completely are melted, they occupy volume at  $(1 - r_{melt}/r_{sol}) \approx 18\%$  higher than in solid state. Therefore, crater's bottom will rise at  $dX_{melt} = 0.18 \cdot \Delta X_{melt} \approx 360$  m only in result melting of rocks in zone II. Additional lift of crater's floor will occur due to increasing volume molten rock by energy  $\Delta E_{melt} = 1.92 \cdot 10^{19}$  J. Increasing volume magma in result this heating may be calculated like:

$$\Delta G_{(II)} = b \cdot S \int_{X_{vap}}^{X_{melt}} T(X) dX = b \cdot T_{vap} \cdot S \cdot X_{vap} (1 - X_{vap}/X_{melt}) = 6.35 \cdot 10^{13} \text{ cm}^3 \quad (12)$$

where:  $T_{\text{vap}} = q_{\text{vap}}/C = 47500 \text{ K}$  – temperature vaporized rocks.

This heating relocates upper boundary igneous chamber as far back at  $dX_{\text{heat}} = DG_{(\text{II})}/S = 227\text{m}$ . Summation three effects shows, that the melt level in crater may grow relative to its original position at  $dX_{(\text{III})} + dX_{\text{melt}} + dX_{\text{heat}} \approx 660 \text{ m}$ , that exceeds of initial depth crater  $X_{\text{vap}}=630 \text{ m}$ . Thus crater completely leveled in result filling melt rocks from igneous chamber located below. At this on modification stage, magma melt may pour out to terrestrial surface.

Duration this phase may be defined by time  $t$  reach thermal equilibrium in system by conduction. The value of  $t$  in order of magnitude can be estimated by formula

$$t = X^2/c \quad (13)$$

where:  $X$  – size of heat-generating layer,  $c$  – factor thermal diffusivity equal  $0.01 \text{ cm}^2/\text{s}$ .

Assuming that heat-generating layer presents itself igneous chamber with size of  $X_{\text{melt}}=2.6\text{km}$ , from (16) we find  $t = 130,000$  years. And total duration of phase modification can be estimated as  $t \approx 3t = 400,000$  years.

### 3.3. Fall large comet

Previous calculation was performed for comet relatively small mass  $m_k = 1.4 \cdot 10^{13} \text{ g}$  with energy  $E_0 = 1.4 \cdot 10^{21} \text{ J}$ . However, mass nuclei galactic comets varies in the range from  $10^{12}$  to  $10^{17} \text{ g}$ , and their energy from  $10^{20}$  to  $10^{25} \text{ J}$ . Analysis shows [2] that galactic comet different size are completely destroyed in atmosphere and, therefore, fall to Earth's surface in form of high-speed gas jet. However in contrast to small comets, objects of large size create in lithosphere a number of new effects that need special discussion.

With using the same formulas, we consider consequences of fall to Earth galactic comet by diameter  $d_k = 3 \text{ km}$ , that possess by three orders greater mass  $m_k = 1.4 \cdot 10^{16} \text{ g}$  and energy  $E_0=1.4 \cdot 10^{24} \text{ J}$ . As before we assume that comet converts at gas jet of length  $a = 1 \text{ km}$  and area of cross section flow  $S = pd_k^2 = 2.8 \cdot 10^{11} \text{ cm}^2$ . In result density jet at the time collision with surface will be  $r_0 = m_k/(pd_k^2 a) = 0.5 \text{ g/cm}^3$ . All other terms remain the same as in problem for comet nucleus with a diameter  $d_k = 300 \text{ m}$ .

Substituting these parameters in calculated formulas, we are also get zones evaporation, melting and heating rocks. Only now their size will be 10 times greater and mass and energy heated rocks in  $10^3$  times larger than for comet diameter of 300 m. The depth crater will be  $X_{\text{vap}} = 6.3 \text{ km}$ , at this from crater will be ejected mass substance  $M_{\text{vap}} = 5.8 \cdot 10^{17} \text{ g}$ , which will fly off with speed  $W = 70 \text{ km/s}$  like in case of small comet. Lower boundary zone molten rock descends at depth  $X_{\text{melt}} = 26 \text{ km}$  and boundary zone heating rocks to  $100^\circ\text{C}$  to depth of  $X_{100}=138.5 \text{ km}$ .

The last value corresponds to depth mantle. Therefore we can not neglect temperature increasing with depth as we did before. We introduce a number changes and additions into the new calculation (fig.4).

The first addition consists in accounting of the asthenosphere. Asthenosphere is partially molten layer substance which underlay relatively cold and hard rocks of lithosphere.

According to modern concepts [13] layer asthenosphere exists no everywhere but where it present, its power strongly changing. It varies from a few dozen to two or three hundred miles. Depth this layer also has strong variability. The roofing asthenosphere beneath continents lies at depth of 80-100 km, and under ocean floor at depth of 50-70 km, rising in some places up to 5 km. In fig.4 shown typical for continental lithosphere situation where asthenospheric layer begins at depth of  $\sim 100 \text{ km}$ .

The second change is more exact accounting a heat of fusion rocks. In contrast to previous calculation, where in the capacity heat of fusion rocks was taken value  $I_{\text{melt}} = 3.4 \cdot 10^3 \text{ J/g}$  for

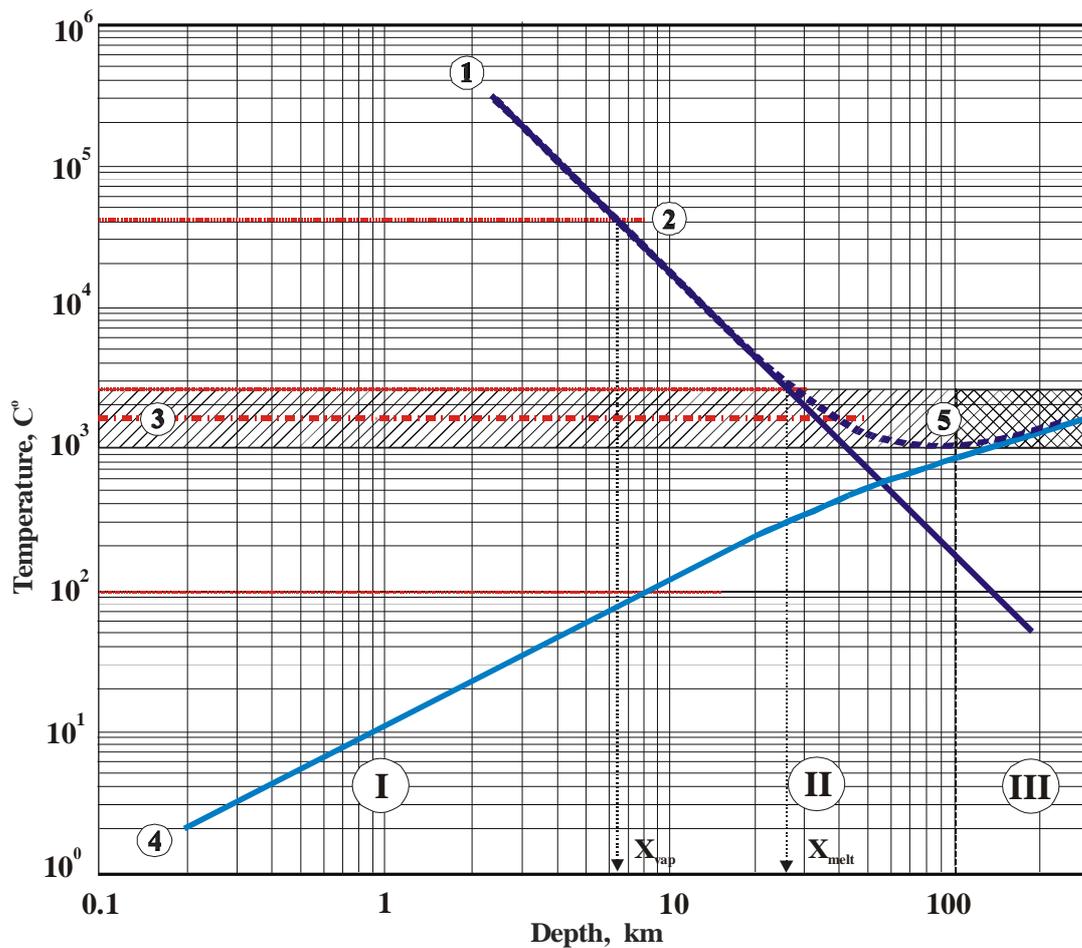


Fig.4. Heating rocks by shock wave created by comet with diameter of 3 km. Legend: 1 – calculated temperature dependence produced by shock wave, 2 – evaporation temperature of rocks, 3 - area of melting rocks (shaded); chain line shows the average temperature for the this area in 1750°C, 4 – natural increasing temperature of lithosphere rocks, 5 – resultant heating temperature of rocks. Notations of zones rocks: I – evaporation, II – melting, III – asthenosphere.

most refractory rocks, now we are consider typical rocks, lying on border crust and upper mantle. The melting point these rocks vary in range from  $\approx 1000^{\circ}\text{C}$  to  $2500^{\circ}\text{C}$ . This range temperature characteristic for asthenospheric layer in general, is shown wide horizontal stripe. Its average temperature  $1750^{\circ}\text{C}$  is according to chain line. In this temperature range are widespread processes of chemical differentiation of rocks caused by phase transformations lherzolite [13]. When temperature increases lherzolite can differentiate in eclogite with density of  $3.45 \text{ g/cm}^3$  and gabbro with density  $2.87 \text{ g/cm}^3$ . Gabbro moves up, increasing Earth crust thickness from below, and eclogite submerges into the mantle.

Finally the third important addition is connected by the natural increase in temperature with depth. It is known that this growth of temperature under continents occurs more steeply than under oceans [13]. Overall, however, both continents and oceans at depths of 100 km temperature rocks is of the order  $1000^{\circ}\text{C}$ . Fig.4 shows a typical temperature rise with depth for continents. According to this curve, at depth of 100 km, adopted as roofing of asthenosphere, temperature of rocks is  $1200^{\circ}\text{C}$ .

Calculations (fig.4) indicate that thermal effect caused by shock wave produced large comets is superimposed on natural growth curve of temperature with depth. As a result the total heating temperature reaches a values lying in area melting of rocks. Therefore if for small

comets, zone II was relatively small and zone III large, then for big comets zone II reaches asthenosphere, and zone heating rocks III disappears quite.

In this case, energy shock wave  $E(X_{\text{vap}}) = 3.36 \cdot 10^{22}$  J in amount 2.4% of  $E_0$  residuary after crater formation exclusively is consumed by melting rocks and heating magmatic melt. At full melting of column rocks, starting from floor crater  $X_{\text{vap}} = 6.3$  km to boundary asthenosphere  $X_a = 100$  km, volume melt will equal  $G_{\text{melt}} = (X_a - X_{\text{vap}}) S = 2.62 \cdot 10^3$  km<sup>3</sup>. If we assume that average density rocks equal to 3.3 g/cm<sup>3</sup> and resulting melting decreased to 2.7 g/cm<sup>3</sup>, then increase length column due to formation magma reaches of 17 km. The heating magma adds another 3 km. As a result, magma completely fills crater depth 6.3 km, and rest amount in volume of 384 km<sup>3</sup> flows out on surface. Effused substance is  $1.04 \cdot 10^{18}$  g, which is more than 80 times exceed comet's mass.

If rocks are melted not completely, then volume and mass of magma will lower. Speaking of partial melting rocks, we are referring to situation (fig.4) when resultant temperature curve lies inside area melting of rocks. In this case after passage shock wave minerals with lower melting points will pass into magmatic melt in the sequence. Feldspars ( $T_{\text{melt}} = 1100 \div 1550^\circ\text{C}$ ) will pass into melt in the first queue, pyroxenes ( $T_{\text{melt}} = 1300 \div 1550^\circ\text{C}$ ) will follow them, and olivine will melt ( $T_{\text{melt}} = 1600 \div 1800^\circ\text{C}$ ) in last queue. Naturally on final stage process these minerals begin crystallize in reverse sequence when temperature melt reduces.

Obviously, the process modification rocks, heated by large galactic comets take more time than for smaller ones. Moreover formula (13) which we have used to estimate length of time modification process in case of small comets, for column rocks by length  $X_a \approx 100$  km is not applicable. However, if as value  $t$  we will understand time transfer heat by conduction through side surface of magma column into surrounding rocks, then obtain estimate of life time column  $t \approx 3t \sim 2$  million years. In absence intense flow of magma along column, system splits during this time on series smaller igneous chambers.

Since 2/3 surface Earth is covered by water, we also will estimate effect fallings galactic comets into ocean. We take water density  $r = 1$  g/sm<sup>3</sup>, and its heat evaporation  $q_{\text{vap}} = 2.26 \cdot 10^3$  J/g. Calculations by the above formulas give for comet diameter of 300 m, layer evaporated water 10.5 km, and for comet with diameter 3 km of 105 km. Accepting an average depth ocean 3.7 km, energy evaporation water in the first case is 13.5% and in the second 1.35% of  $E_0$ . Therefore from energy point of view we can neglect energy loss of jet on evaporation water during formation crater on ocean floor. However appearance in water and ocean floor of deep channel which quickly fills with water will generate number of new physical processes that may affect our estimates. These processes are sufficiently complex and claim special modeling. Therefore prior holding such modeling the issue left open.

Another thing, when asthenospheric layer lies no deep under Earth's surface. In particular, in axial zones mid-ocean ridge (MOR), especially on East Pacific Rise, roofing asthenosphere is located at depth of 3-4 km. [13]. As the distance from MOR, roofing asthenosphere slowly sinks and for most oceans its distance from surface is 50-70 km. In such conditions even small galactic comets can create channels molten rock, which reach asthenosphere. Through these channels magmatic melt from asthenosphere can rise to the top, and then either flows out at ocean floor or crystallizing before reaching surface. Classification geological bodies, arising in these processes we propose in [14].

#### 4. Discussion of results

Central to the article takes calculation of thermal effects, which galactic comets can create in lithosphere's rocks. We assume that in Earth's atmosphere these comets are converted into gas jet from vaporized cometary material and shock heated air, which gets energy and momentum

of comet. As result of inelastic interaction of jet with Earth's surface, energy and momentum of comet are passed into lithosphere. With attraction of the hydrodynamic theory of high-speed impact of Lavrentiev [9], was showed that in this case is arise narrow-beam shock wave, which propagates deeply in lithosphere and causes thermal changes of column of rocks length  $\sim 10\div 100$  km.

Using the theory of Lavrentiev we performed calculations of physical effects for galactic comets two sizes: "small" –  $d_k = 300$  m ( $E_0 = 1.4 \cdot 10^{21}$  J) and "large" –  $d_k = 3$  km ( $E_0 = 1.4 \cdot 10^{24}$  J). In both cases the diameter of cylindrical column rocks, heated by shock wave was considered equal of  $\approx 2d_k$ . Our calculations showed that top part of column rocks evaporates by heating. The depth of crater roughly corresponds to its diameter. The total mass of evaporating rocks is in  $\sim 50$  times more that comet's mass In this case on the evaporation of rocks and, as a consequence, the crater's formation goes 97.6% of initial energy  $E_0$  of shock wave. The residuary part of energy wave  $\sim 2.4\%$   $E_0$  is spent on heating and melting of rocks under crater's bottom. The zone rocks which have tested complete melting (igneous chamber) at fall small comet is equal 2 km, and total length column of heated rocks reaches of 14 km (fig.3). Lengths of zones evaporation (I), melting (II) and heating (III) of rocks are correlated with comet's diameter  $d_k$  like 2:6:40. Formation whole structure is determined by velocity shock wave and is equal  $\sim 0.1$  s.

But this is only the first phase of process. On the next modification phase, which requires time  $\sim 400$  000 years, there is a redistribution of heat between rocks, variously heated by the shock wave. At this stage of the crater is filled with molten rock from the zone II and actually ceases to exist. The quantity magma may exceed volume of crater and some its part outpoured on surface.

For large comets very important role is played existence of asthenosphere as well as temperature gradient. Both factors which we didn't took in account for small comets leads to fact that column of rocks, heated to melting temperature reaches asthenospheric layer (fig.4). As a result channel arises through which magma can rise up from asthenosphere to terrestrial surface in quantities far exceeding the volume igneous chambers, created by galactic comets. Obviously that channel by long in many tens kilometers and in diameter of  $1\div 6$  km, connecting asthenosphere with surface can not exist for a long time. The lifetime of the channel if magma is no moved along channel, according to our estimates, is  $\sim 2$  million years. During this time thermal energy of magma through walls of the channel is passed to environment, which causes heating and partial melting of rocks. As a result cylindrical magma column splits on system of smaller igneous chambers "pop-up" to surface. In the process lifting, these cameras are deformed, and their rocks are experiencing decompression heating and magmatic differentiation. If the thermal energy is not enough then magma crystallizes, forming different in shape and composition intrusive bodies [5, 19].

As the heating temperature decreases with depth the magma in upper part of the melting zone has more refractory minerals than at bottom of the igneous chamber. This circumstance can explain by existence two major types of oceanic basalts: tholeiitic and alkaline. It is known [13] that tholeiites are usually confined to areas with extremely high temperature gradients (up to  $100^\circ\text{C}/\text{km}$ ) and are formed at partial melting rocks at shallow depths. Whereas alkali basalts are generated at great depths (typically 50-100 km) in areas with lower temperature gradient ( $<30^\circ\text{C}/\text{km}$ ) and are characterized by a lesser degree melting of mantle rocks.

Our calculations can answer also other important geological issues. For example, consider the problem formation of seamounts and hot spots [2, 7]. Seamounts are called isolated volcanic structures of conical form height of over 500 m. Number underwater volcanoes on our planet is huge. Only in the Pacific their number is about 130,000. In whole seamounts occupy of  $\sim 6\%$

10% surface of ocean floor [17]. At the mid-ocean ridges the mountains is very small. Away from MOR, i.e. with increasing age bottom, number and size of seamounts systematically increasing. The largest seamounts height of 5-6 km and volume of  $\sim 10^3 \text{ km}^3$  meet only in deep-sea trenches. This pattern is characteristic for seamounts in all oceans.

According to geophysical data [18], many seamounts have below a igneous chambers (sometimes more than one) of size tens kilometers. These chambers are usually located at depths of  $\sim 10$ -50 km and are associated with seamount by channel diameter of 1 km. Magmatic melt from this chamber pours out to ocean bottom, causing growth of underwater mountain. The degree melting of rocks camera is 10%. In the case of volume chamber  $10^4 \div 10^5 \text{ km}^3$ , melt volume is  $\sim 10^3 \div 10^4 \text{ km}^3$ , which is enough to create a very large seamounts. Lifetime igneous chambers are  $\sim 100$  million years, so the process eruption of magma can last all of 180 million years existence of modern oceans.

Estimates show [2] that average on sizes galactic comets are quite capable to creating seamounts height of 5 km. In the case when igneous cameras arise comparatively close to surface, magmatic melt usually reaches ocean floor that leads to growth seamount. In zones of spreading ocean-floor velocity horizontal movements of oceanic lithosphere is maximum for the surface layer of chilled oceanic plate and lower for underlying mantle rocks. So when the igneous chamber locates on shallow depth and moves with the plate, lava is forming solitary seamount which grows, keeping the cone shape in process spreading ocean-floor. If velocity movements of ocean floor more against igneous chamber, the seamount in process of growth obtains asymmetric shape, characteristic to many submarine volcanoes. And, finally, if the camera is so deep that it doesn't move when oceanic plate motion, there is turns out the effect "hot spot", when instead of single volcano appears volcanic ridge or a chain of seamount which stretches sometimes, as in Hawaii, by 2500 km.

The present results indicate that hot spots can be created only largest comets, which produce permeable channels that reach asthenosphere. Along these channels magma lifts to surface in quantities required for hot spots and trappean eruptions. This hypothesis, in our opinion, is more acceptable physical explanation of origin hot spots and mantle plumes than other present ideas [19]. Moreover, it is well supported by petrological studies [20]. These studies show that relocating magma to surface occurs within very narrow and long igneous columns of length  $\sim 10$ -100 km. Column diameter is estimated to be  $0.6 \div 7 \text{ km}$  and the rate of rise of magma in them, depending on the viscosity is from  $0.3$ - $0.5 \text{ m/year}$  to more than  $3 \text{ m/year}$  [21].

According to modern concepts, basaltic melts are generated at depths of  $100 \div 230 \text{ km}$ . After that they relocate up the narrow channels to surface and, starting from depths of  $\sim 20 \div 30 \text{ km}$ , can form a chambers in which magma crystallizes in form different intrusive bodies. The formation processes of intrusive and extrusive rocks go beyond the model that we use, and in this paper are not discussed. These issues of petrology covered in the relevant literature [16, 20]. Let's pay attention to another important result of this work. Calculations for the comet with energy  $E_0 = 10^{25} \text{ J}$ , seven times higher than the one for which plotted dependence (fig.4), show that shock waves these comets can heat rocks at depths  $\sim 200$ - $250 \text{ km}$ . These depths actually coincide with observed lower boundary of asthenospheric layer [13]. Earlier [22] we have assumed that the bombings of Earth by comets of the Galaxy, repeating across  $20$ - $37$  million years, are causing origin of asthenospheric layer our planet. Coincidence depth of heating rocks by cometary shock waves with lower boundary asthenosphere is an additional argument in favor of this hypothesis.

### Key findings and conclusions

1. The two-stage mechanism of interaction galactic comets with Earth is substantiated. At the first stage galactic comets are destroyed in atmosphere, forming high-velocity gas jet from

mixture vaporized cometary material and shock heated air. In the second stage this jet forms super-directional shock wave that penetrates deep into lithosphere and passes its energy at heating rocks in igneous chambers.

2. Processes of second stage we investigated on basis of hydrodynamic theory developed by Lavrentiev to study high-impact processes. Calculations showed after passage shock wave is formed cylindrical column rocks, heated to different degrees. In general case column has three zones: of evaporating rocks – crater, their melting – igneous chamber, and heating zone rocks to temperatures below melting point. During stage of modification by duration of 0.4-2 million years, there is a redistribution heat along column, crater is filled molten rocks from igneous chamber and virtually disappears. Large comets form zone melting, which reaches asthenosphere. At this is forming through-channel through which magma from asthenosphere comes up. In the case of lava outpouring on surface ocean floor are formed seamounts and hot spots, and at crystallization lava under surface are formed different types of intrusive bodies.

3. The calculations confirm the author's views [2], according to which fallings galactic comet play important role not only in processes appearance craters on Solar system planets, but also in tectonic-igneous processes on Earth. Presented in this paper new results brings us to problems petrophysics of igneous rock. However evolution igneous chambers, as well as melting and crystallization rocks under influence cometary shock waves go beyond the Lavrentiev's hydrodynamic theory. In this work these problems are considered only marginally. Obviously they need solved with assistance other theoretical models.

To summarize this work, it must be said that it permit suggest new approach to explanation of a number fundamentally important to modern geology phenomena and processes related to reasons formation igneous chambers, seamounts, hot spots and plumes. The questions origin these phenomena are actively debated last half century [19].

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## About models of continuum mechanics

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The purpose of the present study is to clarify the description of the mathematical model of continuum mechanics. At present for continuum mechanics the formulation of equilibrium force conditions are used. It gives us symmetric pressure tensor and disturbance of continuum medium. This work is development of more full mathematical model for continuum mechanics and rarefied gas. It is devoted to the influence of consideration an angular momentum variation in an elementary volume and cross flux on it's sides. Taking into account the angular momentum law nonsymmetrical stress tensor is received. The method for calculation of nonsymmetrical part is suggested. The equations for gas are found from the modified Boltzmann equation and the phenomenological theory. For a rigid body the equations are used of the phenomenological theory, but changed their interpretation. For rarefied gas the second term in collision integral of the Boltzmann equation is taken into account to calculate the self-diffusion and thermo-diffusion that was foretold by S.Wallander. The Hilbert paradox is solved. We discussed the problems that can be appearing for consideration the angular momentum variation in an elementary volume near the surface and into boundary layer. Conditions of the existence A.N.Kolmogorov inertia interval are established. Conjugated conditions at surface without the Knudsen layer are written to count friction and heat flow to the surface. Conditions of influence of angular momentum are discussed. The examples are given [1-55].

### 1. Introduction

Mechanics is one of the oldest sciences (about three thousand years). Its emergence and development is related to practical needs. This paper will focus on the mechanics of continuous media, although it remains at the heart of the movement, the sustainability movement and interaction of media influence in the motion, that is, those questions are traditionally engaged in the university and its founder academician N.G.Chetaev. It seems that the continuum mechanics, evolving over many centuries, has made significant progress in the formulation of the fundamental laws, but it is not. Currently used by the equations of motion, in our view, derived from the insufficiency of the physical concepts and must be replaced. The need for revision of the theory say the observation in solid mechanics in porous and composite materials, the modern theory of nanostructures, celestial mechanics, fluid mechanics, gas and plasma. A large number of experiments tell us about the bad turbulence theories, bad equations of state of water and other liquids, bad boundary conditions in the case of transient flow, bad model for various structures in the solid, if gradients and large cracks [1-12, etc.] that is, the importance of the gradients of the physical quantities (density, momentum, energy, angular momentum). For example, twisting the ends of nanofilms, thin stream of liquid and gas, some heat issues, the interaction of atoms in the crystalline body in steel, the interaction of the gas atoms and the surface are bad investigated.

Not determined the effect of the collective effects in the interaction potentials of molecules. We have not moved in the theory of determining the structure of shock waves, especially for structural gas. Known numerical solutions are received on the basis of the classical Boltzmann equation but in the initial stage of relaxation and excitation as a result of metabolic processes of the excited atoms is too small and can't be used by statistical laws. In our opinion, require special attention findings from experimental studies carried out on the old technology. Older results are not very accurate, even for well-known model and model problems because the study used outdated technology. Awareness will be in need to review. Results will be determined from the results of using non-destructive methods. This is particularly evident for the boundary layer and the outer boundary and to determine the structural features of the solid. Costs would be paid off quickly. Execution of works of the first type is realistic to modern technology in order to verify the proposed theories. Old theories, like the old science, based on the close consideration of selected items. In Fluid Mechanics this led to neglect of the processes within the volume and placement of the axis of

rotational inertia inside the elementary volume, which led to the rejection of the effect of rotation of the elementary volume. “Classical Mechanics in XX century developed very rapidly and has undergone significant changes. The major change associated with the transition to the foundation of the Eulerian mechanics, a natural development of Newtonian mechanics. With all that has been achieved before, is preserved and does not require any changes. The main difference from the Eulerian mechanics Newtonian mechanics is a more complete and consistent record of spin moves. The balance equations of momentum and angular momentum are independent laws. The concept of legal force, but the time is not completely determined by and is an independent entity” [13]. Elementary volume, as an open system, exchanging components of all physical quantities in all directions. It is unlikely that you can use to describe the current tube turbulence. These issues are discussed, but working on them by individual researchers. The exception is, perhaps, the Tomsk group (V.E.Panin [14, 15]. Issue of more detailed study of the continuity equation set V.A.Bubnov [16] Analysis of the conservation laws of fluid mechanics is given in T.G.Elizarova [17], Yu.V.Sheretov [18], V.M.Ovsyannikov [19].

In fact, the same issue dealt with by B.N.Chetverushkin [20]. It seems far-fetched question about the irreversibility of processes at the time reversibility of the equations of motion of Newton. In the numerical simulation of the small perturbations are always present, in the physical experiments always make a big role, collective effects, playing the role of small perturbations. For example, the slightest violation of the homogeneity of the medium creates angular momentum. No matter how small its value may be, the final state is different from the state dictated by the initial data, and Newton's equations. Consequently, under time get another task (for the other primary source of data). These are only some of the problems that need to be reviewed and the development of theory. These problems encountered in my work on real projects and is also identified concerns research. Next, we discuss some issues of continuum mechanics, which we managed, we believe, to clarify.

At present for continuous mechanics formulation of equilibrium force conditions are used. These give symmetric pressure tensor and disturbance of continuous medium that require from us to perform some addition conditions.

The law of equilibrium force ensures conservation laws for masses, liner momentum, energy, angular momentum.

The law of conservation for angular momentum is basic too. In classical case the angular momentum does not build and it degenerates to symmetric pressure tensor. L.D.Landau, C. de Groot, P.Mazur, I.Deyrmati, I.Prigogine, L.I.Sedov, A.A.Ishlinskii, D.D.Ivlev et al. point at the role of angular momentum. It is the most developed area in the momentum theory of elasticity (R.D.Mindlin, V.A.Palmov, A.G.Gorshkov, E.I.Starovoitov, A.V.Yarovoy, V.Levin, S.E.Kanaun, E.L.Aero, etc.). Any movement of an elementary volume of liquid at the moment can be considered as a result of the following motion: quasisolid motion that is translation with selecting pole, rotating motion around this pole and deformation motion. This theorem was proved by H.Gelmholtz. L.Prandtl formulated conception of hardplastic body as the theory of ideal plasticity. Usually we do not take into account twist velocity. The angular momentum is responsible for the twist velocity. Thus, in the classical Newton mechanics we have four conservation laws: masses, liner momentum, energy, angular momentum. In continuous mechanics we use only three first laws. Last law degenerates in symmetric stress tensor  $P$ . The kinetic theory does not save the situation. The law of angular momentum does not implement in the Boltzmann equation. Classical theory the second viscosity is predicted but usually we supposed that it is need to take into consideration for molecules with inner of

degree of freedom or for dense gas. The modified kinetic theory gives the second viscosity for molecules without structure.

In the Navier-Stokes equations and boundary layer the non-linear and dissipation interact among themselves. Nonlinear lead to distortion of initial signal in result of moving harmonic with it's velocity and lead to stratification disturbances on separate harmonics. Dissipation decreases the signal amplitude. But it is known that besides enumerated factors for some tasks dispersion effects are essential as independence value. Classic equation for which presence of nonlinear, dissipation and dispersion are essential is Korteweg-de Vries-Burgers (KBGB) equation. It is well known that dispersion leads to mixing separate harmonic. The signal form is modified. If term with the third derivatives is added to the system of the Navier-Stokes equations they change to KBGB equation (for velocity). The modified equations of aerodynamic follow from the modified Boltzmann equation and were received for the particles without structure. The angular momentum does not contain the new dimension constants. So the similar solution for classic equations are similar for modified equations. For elasticity theory equations did not change but were proposed another interpretation and added the angular momentum equation to the classic equations. Taking into account the angular momentum law nonsymmetrical stress tensor is received. The method for calculation of nonsymmetrical part is suggested. Now for consideration of angular momentum the theory of brothers E.Cosserat, F.Cosserat and their modifications are used. Their theory contains additional constant with dimension of length that determined from experiments.

Pass from the integral form to the differential form can be, for smooth functions, suggesting that the contraction of the point and non-registration of any processes within the volume. The theoretical part of the conservation laws are obtained by writing the balance equations for the volume element, located in the infinite space. For each conservation chooses its elementary volume [21]. Earlier contributions since ignored on the grounds that the volume an order of magnitude smaller than the surface. Accounting momentum with solid deformation is performed using the equations for the calculation of the moments of a non-symmetric part of the stress tensor. It can be shown that the functional with the momentum for elasticity problems is a local minimum. The estimates of the contributions of surface and bulk quantities is a local minimum is reached, the global is the same. Furthermore, in the classical theory considered only the convective flow as a portable size. Besides usually only convective flows the cross flux performed. As example for density was take into consideration only the flows of density across the surface of an elementary volume. Conclusion of the equation with the more full physical idea as classical was done by S.V.Vallander in 1951 year (for the cross effects) [22, 23] without including angular momentum. The self-diffusion, thermo-diffusion and diffusion for linear momentum were considered for the perfect continuous gas in supposition about internal energy  $E = c_v T$ . Here  $c_v$  is coefficient of thermal capacity under constant volume. For great gradients of physical values colliding molecules are belonging to different distribution functions of the Boltzmann equation can have different macroparameters (density, temperature, velocity) in the Boltzmann equation. To take into consideration the influence of the distribution function variation over times by the mean time  $t$  between collisions of particles was suggested by B.V.Alexeev [24]. We suggest that the second term R.Taylor series in collision integral necessary to take into account. It received after expansion the distribution functions in collision integral over space coordinates. Differentiating the equilibrium distribution function of collision integral the structure of formulas by S.V.Vallander can be received for interacting molecules with hard potential. In other case it is necessary to take averaged cross section. In general case (non-stationary) the collision integral can be written in form suggested by B.V.Alexeev. Constants are defined by the species of

concrete potential. We have to use the average time between collisions, which is inversely proportional to frequency, despite its individual character for the two colliding molecules. For numerical solution can be used the average values of collision integral for concrete velocity. It is essential that the local-equilibrium distribution  $f_0$  the bulk of the solution of the Boltzmann-Enskog method Chapman contains macro parameters, calculated from the Euler equation [25-28]. Formal macro parameters are calculated with zero-order, although built distribution function of the first order. This fact was noted by G.Gilbert without the permission of the paradox. The order of the new equations (for the density and linear momentum, energy) more than in the classical case. If we are dealing with a continuous medium, the boundary condition at the outer edge of the boundary layer can be defined by the value of the vortex or the value of the vertical velocity. On the surface you want to set the friction. For the turbulent layer is usually given friction [21, 29]. For a rarefied gas flow must be set apart from the usual boundary conditions. In general, the formulation of the boundary conditions requires further study. Adhesion conditions hold true on a large scale for the flow of Newtonian type. But it is well known that for a small scale, the liquid adjacent to a solid surface, it can slide, and for the very small size of slip becomes significant, that is adjacent to a body fluid does not slip stationary. How does this slip is not clear. In [30] on the basis of a study by molecular dynamics (MD) and the analytical method was proposed two mechanisms of sliding. Both types of slip observed in molecular dynamics, describes the dynamic model, in which each atom experiences a fluid force on neighboring atoms, liquid boundaries covered the overlying fluid, and inhibited surface. So, we have no theory to describe the interaction of the gas with the liquid and the solid in the whole range of parameters. At present, the theory describing turbulence is unclear despite the large amount of work. Usually use the following theory: a direct numerical method, the large eddies, the Reynolds-averaged equations and others. These methods are the main computing tool in areas where a significant role played by local fluctuations in the parameters of gas (unsteady aerodynamics, aero-acoustics, combustion, etc.), and you can use a common approach A.N.Kolmogorov.

In the previous papers we investigated numerically and analytical methods the modified Blasius problem, the Falkner-Skan problem, the flow near infinite plate. For the last case the motionless thin film was detected and the Prandtl formula was received. It has been shown that longitudinal velocity profile for these problems has fluctuations near upper boundary. For the boundary layer turbulence is beginning here. Essentially that the tasks about the flow in tubes, flow among two infinite plates are formulated in our text book as hydrotechnical tasks (middle change is constant). Infinite force is need to ensure this model. Transition streams are sensitivity to small indignations, so it make difficult to receive identical boundary conditions from one experiment to another.

Logarithmic profiles for infinite plates and pipes, in general, are obtained for the product of finite length, although quite elongated. The theoretical profile is defined by introducing the Reynolds stress. The velocity distribution in the laminar part of flow  $u = \frac{t_w}{m}$  and

$m \frac{du}{dy} = \text{const} = t_w$ . For turbulent flow stress of the turbulent wall shear  $t = -\overline{ru'v'}$  is zero.

The Reynolds equation is  $\frac{d^2u}{dy^2} = \frac{dt}{dy} = 0$ . If the stress of friction is the Prandtl formular

$$t = rl^2 \left( \frac{du}{dy} \right)^2 = t_w, \quad l = ky, \quad k = \text{const} \quad \text{to} \quad u = \frac{1}{k} \sqrt{\frac{t_w}{r}} \ln y + C, \quad C = \text{const}$$

Thus we used some suppositions and we solve do not the Navier-Stokes equations and do not solve the equations of boundary layer but the Prandtl equations. Starting equations do not satisfy as on external boundary as on surface. Infinite force is need to ensure this model. The problem of the interval N.A.Kolmogorov [31] region in boundary layer is analyzed as linear velocity can be only near surface but there the velocity is small; so this region far from surface. Our model tell us about logarithm profile for boundary layer for inertial region.

All results are obtained for structure less particles. For structural gas equation for the need to build momentum substantiated by the fact that averaging the potential interaction of the molecules in the angle to calculate the cross section leads to a loss in the process of change of the moment of collision. It seems likely that the best result is obtained by averaging the cross-section. Often, the solution of the problem, examine the stability of the solution to a small perturbation. Given the order of magnitude (surface and volume), it is theoretically possible to conclude on the importance of the contribution of forces created by the volume effect since, based on the study of stability.

The most important issue is the question of modern aerodynamics gas interaction with the surface of the aircraft, but the interaction is poorly understood for transient flow of rarefied gas. Refined formulation of the boundary conditions on the surface of the body and a model of the origin of the surface crack. Background to the proposed method for the near free molecular flow is associated with the work of J.Willis, M.Lunts, D.Lyubonskogo, V.S.Liu [32]. Willis believed that near the solid surface, each molecule collides with another only once and that the colliding molecules belong to different "classes". In other words, it is assumed that most likely the collisions involving molecules undisturbed flow incident on the body, and the molecules are diffusely reflected from the surface. To ensure accuracy, invited to do iteration. Molecules diffuse reflection expected. Analytical method and by cumbersome and daunting not spread. For a slightly rarefied gas N.G.Patterson [33] received the slip velocity and temperature jump, which depend not only on the cross, but the longitudinal gradients of temperature and velocity. Assumptions include still the distribution function of the reflected molecules and mixed model (diffuse-reflective). Interaction of rarefied gases with sleek surfaces studied by A.Ryzhov, R.G.Barantseva, F.Goodman, H.Wachmann, A.A.Pyarnpuu, G.V.Dubrovsky et al. [34-38]. All previous methods except works associated with the free-molecule flow, working with the incident and reflected molecules. In the case of free-flow model wondered reflection of molecules (diffuse, specular, mixed). Currently, the development of computational tools and models allow a solid hope for a more rigorous examination of the question (A.M.Krivtsov, E.A.Ivanova, N.F.Morozov, A.I.Melker and others [39, 40]), but they are mostly modeled gas interaction with nanostructures and the interaction of atoms in a solid. Two variants of the calculations. One option involves a direct calculation using the model potentials of interaction between gas molecules and solid-state model, and the second is related to the proposed new equation for the distribution function [28, 41]. Analysis of the equation leads to some general conclusions. In [41] it was suggested that the boundary conditions rather than the study of the Knudsen layer with the characteristic size of the order of the mean free path to consider a layer of a few radii of the molecular interaction. Problem of the interaction with the surface of the gas near the body has the features related to the fact that the ratio of the mean free path of the gas molecules to the distance between the molecules of a solid high. During the interaction of the gas molecules with solid distribution function does not change ( $10^{-13}$ s), but the next interaction molecule belongs to another part of the distribution function. Thus, the interaction will be of unstable character.

The result of the interaction of moving gas molecules with the surface is a diffusion of gas molecules along the surface with subsequent partial accommodation of the energy gain due to the interaction of the molecules with the solid (in the case of high surface temperature). The analysis points to the importance of profiling the distribution function of the incident flux and significant impact on the collective capacity of the surface crack nucleation and the distribution function of the reflected molecules.

Analytical study of the classical weak shock waves to a structure less gas can be found in [42], subject to certain terms in the equation S.V.Vallander can be found in [43]. As for the structure of gas with a threshold energy of destruction, and for an ideal gas can move in the opposite direction at Mach numbers close to unity. So, the modified equations of modified aerodynamics follow the Boltzmann equation. New equations with the inclusion of angular momentum and cross streams contain only two dimensionless parameters: the Reynolds number and Prandtl number. We begin with a discussion of our modified equations, we give a new interpretation of the classical equations of continuum, and then give some examples.

## 2. Equations

### 2.1. The modified equations.

Presented in previous papers the equations of motion, energy and angular momentum were obtained before, but the use of force equilibrium conditions did not require the calculation of angular momentum. The modified equations are

$$\begin{aligned} \frac{\partial r u_i}{\partial t} + \frac{\partial}{\partial x_i} \left( r u_i u_j + P_{ij} + x_i \frac{\partial P_{ij}}{\partial x_i} \right) - \frac{X_i}{m} r &= 0 \\ \frac{\partial}{\partial t} r \left( \frac{3}{2} RT + \frac{1}{2} u^2 \right) + \frac{\partial}{\partial x_j} \left[ r u_j \left( \frac{3}{2} RT + \frac{1}{2} u^2 \right) + u_k P_{kj} + q_j \right] + \\ + \frac{\partial}{\partial x_i} x_i \frac{\partial}{\partial x_j} \left[ r u_j \left( \frac{3}{2} RT + \frac{1}{2} u^2 \right) + u_k P_{kj} + q_j \right] &= 0 \\ \frac{\partial \mathbf{r}}{\partial x} \times \mathbf{p}_x + \frac{\partial \mathbf{r}}{\partial y} \times \mathbf{p}_y + \frac{\partial \mathbf{r}}{\partial z} \times \mathbf{p}_z + x_j \frac{\partial}{\partial x_j} (\mathbf{P}_j) &= M_I \end{aligned}$$

There are  $t$  – time,  $x, y, z$  – coordinate,  $r$  – density,  $P_{ij}$  – stress tensor,  $u$  – velocity,  $q$  – thermal flow,  $R$  – gas constant.

The last equation is to determine the degree of asymmetry of the stress tensor. The question arose when we are writing the law of conservation for density. Try to get it from phenomenological principles. The modified equation for density was received from the

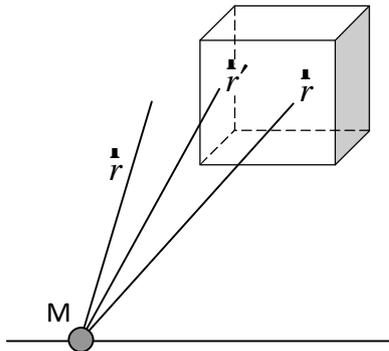


Fig.1.

kinetic theory in the form  $\frac{\partial r}{\partial t} + \frac{\partial r u_i}{\partial x_i} + \frac{\partial}{\partial x_i} \left( x_i \frac{\partial r u_i}{\partial x_i} \right) = 0$ .

The figure 1 shows that the linear velocity  $\dot{\mathbf{u}} = \dot{\mathbf{w}} \times (\mathbf{r}' - \mathbf{r})$  is the velocity with respect to point M for quasi-solid movement around axis  $\mathbf{r}$  without forward velocity. The point M may itself be involved in the rotation. For an elementary volume  $\dot{\mathbf{u}} = \dot{\mathbf{w}} \times (\mathbf{r}' - \mathbf{r})$  formula means a rotation around the axis of centre of inertia but axis of moving of elementary volume can be lie outside it. So we have for twisting an elementary volume.

$$\int_{(s)} (\nabla r u (\mathbf{r}' - \mathbf{r}_c))_n ds = \int_{(s)} \text{div} (\nabla r u) (\mathbf{r}' - \mathbf{r}_c) dv$$

The degree of asymmetric stress tensor we can received from momentum equation (in projections).

$$\begin{aligned} y \left( \frac{\partial s_{xz}}{\partial x} + \frac{\partial s_{yz}}{\partial y} + \frac{\partial s_{zz}}{\partial z} \right) - z \left( \frac{\partial s_{xy}}{\partial x} + \frac{\partial s_{yy}}{\partial y} + \frac{\partial s_{zy}}{\partial z} \right) + t_{zy} - t_{yz} &= 0, \\ x \left( \frac{\partial s_{xz}}{\partial x} + \frac{\partial s_{yz}}{\partial y} + \frac{\partial s_{zz}}{\partial z} \right) - z \left( \frac{\partial s_{xx}}{\partial x} + \frac{\partial s_{yx}}{\partial y} + \frac{\partial s_{zx}}{\partial z} \right) + t_{zx} - t_{xz} &= 0 \\ x \left( \frac{\partial s_{xy}}{\partial x} + \frac{\partial s_{yy}}{\partial y} + \frac{\partial s_{zy}}{\partial z} \right) - y \left( \frac{\partial s_{xx}}{\partial x} + \frac{\partial s_{yx}}{\partial y} + \frac{\partial s_{zx}}{\partial z} \right) + t_{yx} - t_{xy} &= 0 \end{aligned}$$

### 3. Examples

**3.1. Beams problems.** Consider protozoa task about beam [44-46]. Closer definition of equilibrium will be obtained if angular momentum would take into attention.

#### 1). Beam

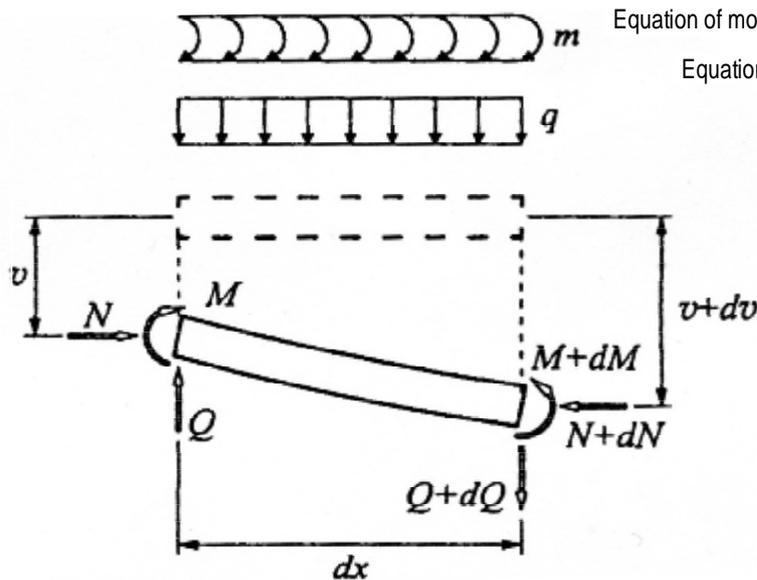


Fig.2. Elementary area of beam

Equation of momentums  $dM - Qdx - m dx - Ndv = 0$

Equation of force  $dQ + qdx = 0$

$$M' - Q - m - Nv' = 0$$

or  $-M'' = q - m' - (Nv')' = 0$

Now

$$-M'' = q - m' - ((Nv)')' = 0$$

For this problem the solution for general case and without  $N'$  (for  $N=const$ ) is the same. Consequently we can choose the beam of variable cross-section. For beam on elastic foundation results distinguish.

2). For another problem the solutions can be different too. In case of transmission pressure at

beam by intermediate designs of external force  $P$  (fig. 3) passes through hard lever of length  $l$  that can twist around hinge  $H=P/\cos\alpha$ , here  $H$  is normal force in hard hinge.

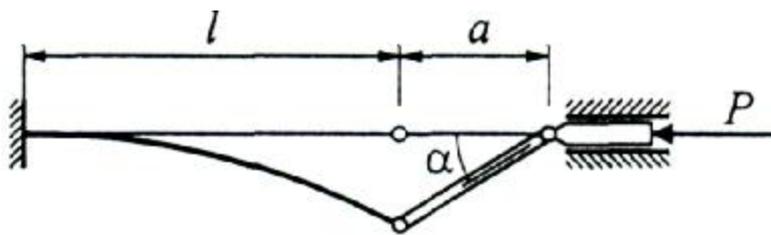


Fig.3. At the end of elastic beam transfers force is  $Q$ .

At the end of elastic beam transfers force is  $Q=H\sin\alpha=Ntga$ . As  $tga=v(l)/a$  at  $l$  we have in classic case  $Q = H \sin \alpha = Ntg\alpha$

$$EIv'''(l) + N \left[ v'(l) + \frac{v(l)}{a} \right] = 0.$$

So we have at end of length

$$v(0) = 0 \Rightarrow C_1 + C_4 = 0, \quad v'(0) = 0 \Rightarrow C_2 + kC_3 = 0$$

$$M(l) = 0 \Rightarrow v''(l) = 0 \Rightarrow C_3 \sin v + C_4 \cos v = 0.$$

$$v'''(l) + k^2 v'(l) + k^2 v(l)/a = 0 \Rightarrow C_1 + C_2(l+a) + C_3 \sin v + C_4 \cos v = 0.$$

Now  $Elv'''(l) + (Nv)'' + \frac{P}{\cos a} \frac{v(l)}{Q} = 0.$

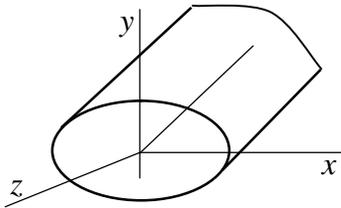


Рис.4 (задачи 1-4)

If we write equations for stress values of elasticity theory we have six unknown quantities and three equations in classical case. In our case we add the equation for angular momentum but we have nine unknown quantities. As a result we have old situation. We suppose that only possible solution we can receive using the classical supposition. Consequently we need to add equations to write the geometrical and physical property.

3). Now consider four educational problems for long beam.

The first is beam under longitudinal force.

Suggested that  $s_x = s_y = t_{xy} = 0, s_z = s_0 = N/e = \text{const}; t_{yz} = t_{zx} = 0.$  There are three angular momentums are equal to zero. So new way a problem do not new results. The second is clean curve. Let  $s_z = ax + by, t_{zy} = t_{zx} = 0; N = 0, M_x = M_z = 0.$  There are two angular momentums are equal to zero. Consequently the laws of forces are fulfill as in this case all three laws of force's balance must be equal to zero. For problem about transverse beam gives same results as for the second case. But for problem about twisting of the primate beam we have another solution  $t_{xy} = f(z)$  besides classic solution ( $s_x = s_y = s_z, t_{xy} = 0$ ). This result we have as two angular momentums at the end  $z=l$  are equal to zero.

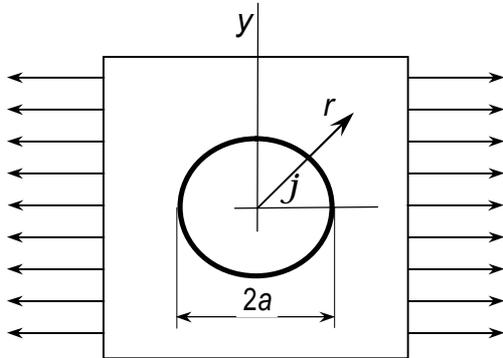


Fig.5 (task 5)

4). Consider the Lamé and Kirsh problems. In the first problem loading on the sides is steady. So momentum can be only in corner. For the same loading on all sides angular momentum does not work, but for different loading momentum work and added strength will be activity. The result is raising ore go down. For the second problem besides the corner's effects we have momentum near the axis  $z.$  So at cross of the circle and axis  $y$  we have added forces that have the direction of the stress on side.

So new way a problem do not new results. The second is clean curve.

**2. Boundary layer.** The Blasius problem was considered in [28, 29] by numerical and analytical. Some results for infinite plate will be formulated here. The equation for this case is

(the new term is selected) 
$$\frac{d}{dy} \left( m \frac{du}{dy} \right) + \frac{d}{dy} \left( my \frac{d^2u}{dy^2} \right) = 0.$$

Boundary conditions are  $y = 0, u = 0, m(du/dy) = t_w; y \rightarrow \infty, u = U_\infty.$

Integrating gives 
$$m \frac{du}{dy} + my \frac{d^2u}{dy^2} = \text{const} = t_w$$

There are  $y$  – coordinate,  $r$  – density,  $u$  – velocity,  $m$  – viscosity. Index “w” is relative to surface. From boundary condition we have  $\text{const} = t_w$  ( $t_w$  is skin friction). Integral of the equation is  $u = C \cdot \ln y + t_w / m \cdot y + \text{const}$

Possible variant to satisfy boundary conditions is that under the  $y = v/v_*$ , where  $v = m/r$ ,  $v_* = (t_w/r_w)^{1/2}$ .

We have  $\ln = 0$ . Later on diminution velocity takes place up zero, derivative can be very large but zero velocity observes between surface and  $y$ . So layer of the rest liquid is formed. Thickness of this layer is  $10^{-3}$  cm. We have not reliable measurements there. Probably for laminar layer there is no layer with zero velocity. For finite body near the edge the gradient of the velocity tends to work. This situation can work into the layer too. Then it works near the rebuilding region. Far from edge friction strives to zero. It does not follow from the theory for semi-infinite plate that the value of the friction is finite but if we suggest zero friction in the first integral we can get the Karman formula for the mixture length. Equality  $t_w = 0$  provides  $u = 0$  as  $y = 0$  and  $u = U_\infty$  as  $y \rightarrow \infty$  and leads to rebuilding of the flow. The profile of the velocity becomes more completed than near the edge. The region with  $t_w = 0$  formulated the inertial layer (N.A.Kolmogorov). Our case relies to logarithm profile for boundary layer instead liner profile for classic. It is interesting that asymptotic friction for half-infinite plate has not the value for infinite plate. In my opinion we have similar situation for tubes. More detailed information for this task and others are in [47-54].

### 3.3. Slip and temperature jump.

For the flow problem, besides the external boundary conditions, should be put adjoint boundary conditions on the surface. The Chapman-Enskog solution of the Boltzmann equation in the interior points of the flow gives a solution asymptotically converges to the solution of the Boltzmann equation for Knudsen numbers tending to zero. However, for any arbitrarily small Knudsen number near the borders of a region in which this series is not a solution of the Boltzmann equation [35]. The thickness of this region is usually considered to be of the order of the mean free path  $\lambda$ . Mean free path does not reflect the specific processes of interaction and inversely proportional to the relative velocity of the collision of molecules. Consequently, the collision cross section of the molecules may depend on the relative velocity. As follows from the analysis, for each speed of the molecule has its own characteristic scale and a gradient of physical variables with a large remainder of the Taylor series expansion near the point of departure of the molecules. Usually set the boundary conditions on the solid wall, in which a solution of the Navier-Stokes equations is the Knudsen layer would coincide with the solution of the Boltzmann equation with the given conditions at the surface. The nature of the interaction of molecules with the surface of the boundary conditions for the distribution function at the lower boundary of the Knudsen layer. We take into account that the mean free path of the gas molecules is much larger range of interaction of the molecules. Select the distance from the surface of the gas boundary of a few radii of interaction between gas molecules and the solid state ( $s$ ). Including molecules in the surface of the number of molecules of the ensemble in the construction of the N-particle distribution function. Let us bear in mind that the mean free path of the gas molecules is much larger selected to calculate the radius of interaction between gas molecules and solid. In general, the equation for the two-particle equation takes the form within a few  $s$ :

$$\frac{\partial f_2}{\partial t} + \sum_{i=1}^2 \left\{ \mathbf{x}_i \cdot \left[ \frac{\partial f_2}{\partial \mathbf{x}_i} \right] + \mathbf{x}_i \cdot \frac{\partial}{\partial \mathbf{x}_i} \left[ x_j \frac{\partial f_2}{\partial x_j} \right] - \frac{X_i}{m} \frac{\partial f_2}{\partial \mathbf{x}_i} \right\} + \mathbf{X}_{12} \cdot \frac{\partial f_2}{\partial \mathbf{x}_1} + \mathbf{X}_{21} \cdot \frac{\partial f_2}{\partial \mathbf{x}_2} + X_{22} \frac{\partial f_2}{\partial \mathbf{x}_2} = 0$$

is possible to look for the solution of aerodynamic problems for transient flow, using the solution of the Navier-Stokes equations, defining the boundary condition at the outer edge of said thin layer through the distribution function (the modified Chapman-Enskog solution) followed by a solution of a layer with the chosen model of the solid. In our view, this approach makes it easier to solve the external flow problem and to more fully account for the

effect of the surface. To test the method solves the problem of determining the influence of the profile of the distribution function of a graduate student A.Shadrin. [55] We investigated the effect of the asymmetry of the distribution function for the equilibrium case. As the directional velocity at the outer edge of a thin layer appears as a slow component, and high-energy. Therefore there is a walk along the surface of the particles to the slow component and the introduction of particles into the crystal for the fast component (fig.10). It should be noted the possibility of a theoretical calculation of fatigue effects ideal surface in normal conditions. To do this, calculated using the equilibrium distribution function of the value of the gas molecules with high energy for a given temperature, with a preliminary study, a similar representation. Calculations confirmed the absence of collisions, the gas inside the layer.

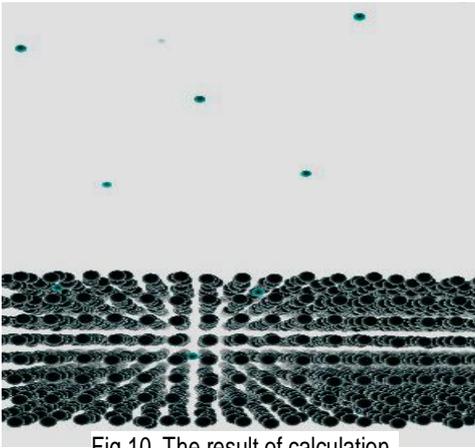


Fig.10. The result of calculation.

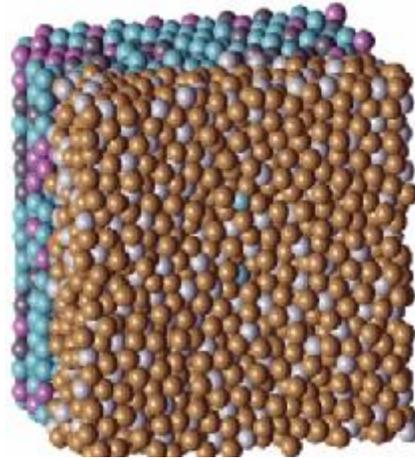


Fig.11. The structure of the surface oxide film.

### 3.4. Shock wave.

The classical structure normal shock waves can be found in [42]

$$ru = r_0 D = a, \quad p + ru^2 + w = p_0 + r_0 D^2 = p_{00}, \quad w = -m \frac{du}{dx},$$

$$\frac{g}{g-1} \frac{p}{r} + \frac{1}{2} u^2 + \frac{1}{r_0 D} (S + uw) = h_0 + \frac{D^2}{2} = h_{00}, \quad S = -k \frac{dT}{dx}, \quad \frac{p}{r} = RT.$$

Here  $r$  – density,  $m$  – viscosity, internal energy  $E = c_v T$ ,  $p$  – pressure,  $u$  – velocity. For modified theory equations were written in [54].

To investigate the weak shock wave for Mach number  $M = 1$  is very difficult. The front of shock wave is not stability for this regime in experiments. The cause of instability is not clearly. In my opinion the cause is in the next. Influence of self-diffusion (only of density) leads to

$$ru = \frac{ma_1}{r} \frac{dr}{dx}, \quad p + ru^2 - m \frac{du}{dx} = p_{00}, \quad \frac{d}{dx} k \frac{dT}{dx} + \frac{4}{3} \left( \frac{du}{dx} \right)^2 + \frac{u^2}{2} \frac{d}{dx} (ru) = 0.$$

The program for computer was done.

## 4. Conclusion

The need to development the theory of continuum mechanics understood. To clarify and supplement the qualitative theory requires new experimental research on modern equipment using non-destructive methods. The theoretical results have to be tested and confirmed by specific calculations. The theory must be based on the principle of equilibrium of moments of forces, which had previously been neglected. Further theoretical analysis should answer the question of the applicability of a particular model for a particular application.

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## **Approximate analytical solutions of the time-fractional coupled Burgers equations and Korteweg-de Vries equations**

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In this article, the homotopy perturbation transform method is used to obtain the solutions of the non-linear coupled Burgers equations and coupled Korteweg-de Vries (KdV) equations with fractional order time derivatives. The fractional derivatives are described in Caputo sense. He's polynomial is used to tackle the nonlinearity which arises in the considered problem. Using the initial conditions, the approximate analytical solutions of the problem are obtained. The numerical procedures reveal that only a few iterations are needed for better approximation of the solutions, which illustrates the effectiveness and reliability of the method. Effects of fractional order derivatives on the solutions for different particular cases are presented through graphs.

### **1. Introduction**

Nonlinear phenomena appear in a wide class of scientific applications such as plasma physics, solid state physics, fluid dynamics and chemical kinetics to name a few. Due to its immense applications in various fields of science and engineering, it has become a popular field of investigation for the researchers from different parts of the world. A wide range of analytical and numerical methods have been used in the analysis of these scientific models. Mathematical modeling of many physical systems leads to nonlinear ordinary and partial differential equations. An effective method is required to analyze the mathematical model which provides solutions conforming to physical reality. Introduction of fractional calculus in nonlinear models has rendered a new dimension to the existing models.

Nowadays, fractional differential equations have gained much attention since fractional order system response ultimately converges to the integer order system response. For high accuracy, fractional derivatives are used to describe the dynamics of some structures. An integer order differential operator is a local operator. Whereas the fractional order differential operator is non local in the sense that it takes into account the fact that the future state depends not only upon the present state but also upon all the history of its previous states. For this realistic property, the fractional order systems are becoming popular. Another reason behind using fractional order derivatives is that these are naturally related to the systems with memory which prevails for most of the physical and scientific system models. Recently fractional calculus has become a popular field of research since researchers from different parts of the world are playing interest to apply the fractional calculus concept in different branches of science and engineering. The main advantage of the fractional calculus is that fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. Hence, great attention has been given to finding the numerical/exact/analytical/approximate solutions of Fractional Differential Equations. The importance of obtaining the exact or approximate solutions of nonlinear partial differential equations in physics and mathematics is still a significant problem that needs new methods to discover exact or approximate solutions. Most of the nonlinear equations do not have a precise analytic solution, so numerical methods have largely been used to handle these equations. Moreover, if the nonlinear systems are of fractional order then it becomes more complicated to solve. Recently many powerful mathematical techniques have been established and developed to obtain numerical and analytical solutions of FDEs, such as Adomian decomposition method [1, 2, 3, 4], Homotopy analysis method [5], Generalized differential transform method [6, 7], Homotopy perturbation method [8, 9, 10], Variational iteration method [11, 12], Modified Laplace decomposition method [13, 14].

Burgers equations are very useful for the authenticating of theoretical and practical applications related to the historical Navier-Stokes equations, since the equations are the second order approximation of the unsteady Navier-Stokes equations. The equations have wide class of applications viz., acoustic transmission and traffic supersonic flows, detecting of shock behavior in fluid flows, measuring the influence of temperature field on the velocity field, describing the incompressible fluid flow coupled to thermal dynamics and also modeling the thermal fluid dynamics of air in building.

The coupled Burgers system is derived by Esipov [15]. The study of coupled Burgers equations is very significant for the reason that the system is a simple model of sedimentation or evolution of scaled volume concentrations of two kinds of particles in fluid suspensions or colloids, under the effect of gravity [16]. Dehghan et al. [17] have applied Adomian-Pade technique to get the solution of coupled Burgers equations. Abdou and Soliman [18] have applied Variational iteration method for getting the solution of coupled Burgers equations.

But Solutions of coupled Burgers equations in fractional order systems are few in numbers. In 2011, Liu and Hou [19] solved the space and time fractional coupled Burgers equations by generalized differential transform method. Recently, Song and Wang [20] have solved the equations using improved version of Adomian decomposition method.

Some types of coupled KdV equations are derived from a two layer fluid model, which are used to describe the atmospheric and oceanic phenomena such as the atmospheric blocking, the interactions between the atmospheric and ocean [21]. But to the best of authors' knowledge the coupled Burgers equations and coupled KdV equations with fractional time derivatives have not yet been solved by any researcher using homotopy perturbation transform method (HPTM).

The Laplace Transform is not suitable for handling the nonlinear equations because of the difficulties that are caused by the nonlinear terms. To overcome these deficiencies, Khan and Wu [22] have proposed homotopy perturbation transform method combining homotopy perturbation method and Laplace transform, which is very effective for dealing with the nonlinear problems. The proposed method provides an approximate solution in series form which converges rapidly. In the present study, the authors have used the HPTM which is concise, accurate and effective for solving the fractional order coupled Burgers equations and coupled KdV equations. Numerical results are obtained for different fractional order derivatives which are depicted through graphs for different particular cases. The beauty of this article can be attributed to the comparison of numerical solutions of probability density functions with the exact one for the integer order system, which clearly exhibits the potential of the proposed method.

## 2. The Homotopy Perturbation Transform Method (HPTM)

The homotopy perturbation transform method is a combination of Laplace Transform, homotopy perturbation method and He's polynomials. To explain the basic idea of the method for solving the fractional order nonlinear non- homogeneous partial differential equation, we consider the following equation

$$D_t^a u(x,t) + Ru(x,t) + Nu(x,t) = g(x,t), \quad 0 < a \leq 1 \quad (1)$$

with the initial condition  $u(x,0) = f(x)$  (2)

where  $g(x, t)$  is a source term,  $N$  represents a nonlinear differential operator and  $R$  is a linear differential operator,  $u(x, t)$  a probability density function and  $D_t^a \equiv \frac{\partial^a}{\partial t^a}$  is a Caputo fractional derivative. Using the formula

$$D_t^a u = \frac{1}{\Gamma(n-a)} \int_0^t \frac{u^{(m)}(x,t)}{(t-t)^{a+1-m}} dt, \quad (m-1 < \text{Re}(a) \leq m, \quad m \in N) \quad (3)$$

where  $\Gamma(\cdot)$  denotes the Gamma function. Taking the Laplace transform on both sides of equation (1) and using

$$L[D_t^a u] = s^a L[u(x,t)] - \sum_{k=0}^{m-1} u^k(x,0^+) s^{a-1-k} \quad (4)$$

we get,

$$L[u(x,t)] = \frac{1}{s^a} \sum_{k=0}^{m-1} u^k(x,0^+) s^{a-1-k} - \frac{1}{s^a} L[Ru(x,t)] - \frac{1}{s^a} L[Nu(x,t)] + \frac{1}{s^a} L[g(x,t)] \quad (5)$$

Operating the inverse Laplace transform on both sides of equation (5), we obtain

$$u(x,t) = F(x,t) - L^{-1} \left[ \frac{1}{s^a} L[Ru(x,t)] + \frac{1}{s^a} L[Nu(x,t)] \right] \quad (6)$$

where  $F(x,t)$  represents the term arising from the source term and prescribed initial condition. Expanding  $u(x,t)$  into power series in  $p$  as

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t) \quad (7)$$

and decomposing the nonlinear terms as

$$Nu(x,t) = \sum_{n=0}^{\infty} p^n H_n(u) \quad (8)$$

where  $p \in [0, 1]$  is an embedding parameter and  $H_n(u)$  is the He's polynomials [22] given by

$$H_n(u_0, u_1, u_2, \dots, u_n) = \frac{1}{n!} \frac{\partial}{\partial p^n} \left[ N \left( \sum_{i=0}^{\infty} p^i u_i \right) \right]_{p=0}, \quad n = 0, 1, 2, \dots \quad (9)$$

Substituting equations (8) and (9) in equation (6), we get

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = F(x,t) - p \left( L^{-1} \left[ \frac{1}{s^a} L \left[ R \sum_{n=0}^{\infty} p^n u_n(x,t) \right] + \frac{1}{s^a} L \left[ \sum_{n=0}^{\infty} p^n H_n(u) \right] \right] \right) \quad (10)$$

Now equating the identical powers in  $p$  we obtain

$$p^0: \quad u_0(x,t) = F(x,t) \quad (11)$$

$$p^1: \quad u_1(x,t) = -L^{-1} \left[ \frac{1}{s^a} L[Ru_0(x,t)] + \frac{1}{s^a} L[H_0(u)] \right] \quad (12)$$

$$p^2: \quad u_2(x,t) = -L^{-1} \left[ \frac{1}{s^a} L[Ru_1(x,t)] + \frac{1}{s^a} L[H_1(u)] \right] \quad (13)$$

$$p^3: \quad u_3(x,t) = -L^{-1} \left[ \frac{1}{s^a} L[Ru_2(x,t)] + \frac{1}{s^a} L[H_2(u)] \right] \quad (14)$$

Proceeding in this manner the rest of the components of  $u_n(x,t)$ ,  $n \geq 0$ , can be obtained, and the series solutions are thus entirely determined. Finally, we approximate the analytical solution of  $u(x,t)$  by the truncated series

$$u(x,t) = \lim_{N \rightarrow \infty} \Phi_N(x,t) \quad (15)$$

where  $\Phi_N(x,t) = \sum_{n=0}^{N-1} u_n(x,t)$ . The above series solutions generally converge very rapidly.

### 3. Solution of the Problems

In this section two examples on nonlinear time-fractional coupled Burgers equations and coupled KdV equations are solved to demonstrate the performance and efficiency of the method HPTM.

#### Example 1.

The nonlinear time fractional coupled Burgers equations are

$$D_t^a u - u_{xx} - 2uu_x + (uv)_x = 0, \quad 0 < a \leq 1 \quad (16)$$

$$D_t^b v - v_{xx} - 2vv_x + (uv)_x = 0, \quad 0 < b \leq 1 \quad (17)$$

where  $0 < x \leq 1$  and  $0 < t \leq 1$

with the initial conditions as

$$u(x, 0) = \sin x \quad (18)$$

$$v(x, 0) = \sin x \quad (19)$$

Taking the Laplace transform of equation (16) and (17),

$$L[D_t^a u] - L[u_{xx}] - 2L[uu_x] + L[(uv)_x] = 0$$

$$L[D_t^b v] - L[v_{xx}] - 2L[vv_x] + L[(uv)_x] = 0$$

Using equation (4) and then applying the inverse Laplace transform, we obtain

$$u(x, t) = L^{-1} \left[ \frac{1}{s} u(x, 0) + \frac{1}{s^a} \{L[u_{xx}] + 2L[uu_x] - L[(uv)_x]\} \right] \quad (20)$$

$$v(x, t) = L^{-1} \left[ \frac{1}{s} v(x, 0) + \frac{1}{s^b} \{L[v_{xx}] + 2L[vv_x] - L[(uv)_x]\} \right] \quad (21)$$

To apply the homotopy perturbation method to equations (20) and (21), we express

$u(x, t) = \sum_{n=0}^{\infty} p^n u_n(x, t)$  and  $v(x, t) = \sum_{n=0}^{\infty} p^n v_n(x, t)$ . The nonlinear terms can be decomposed

as equation (8).

Equations (20) and (21) reduce to

$$\sum_{n=0}^{\infty} p^n u_n(x, t) = u_0(x, t) + L^{-1} \left[ \frac{1}{s^a} \{L[(\sum_{n=0}^{\infty} p^n u_n)_{xx}] + 2L[\sum_{n=0}^{\infty} p^n H_n(u)] - L[\sum_{n=0}^{\infty} p^n H'_n(u, v)]\} \right],$$

$$\sum_{n=0}^{\infty} p^n v_n(x, t) = v_0(x, t) + L^{-1} \left[ \frac{1}{s^b} \{L[(\sum_{n=0}^{\infty} p^n v_n)_{xx}] + 2L[\sum_{n=0}^{\infty} p^n G_n(v)] - L[\sum_{n=0}^{\infty} p^n H'_n(u, v)]\} \right]$$

where  $H_n(u)$ ,  $G_n(v)$  and  $H'_n(u, v)$  are He's polynomials [23] that represent nonlinear terms  $uu_x$ ,  $vv_x$  and  $(uv)_x$  respectively, which are given as

$$H_0(u) = u_0(u_0)_x, \quad H_1(u) = u_0(u_1)_x + u_1(u_0)_x, \quad H_2(u) = u_0(u_2)_x + u_1(u_1)_x + u_2(u_0)_x,$$

$$G_0(v) = v_0(v_0)_x, \quad G_1(v) = v_0(v_1)_x + v_1(v_0)_x, \quad G_2(v) = v_0(v_2)_x + v_1(v_1)_x + v_2(v_0)_x$$

$$H'_0(u, v) = (u_0 v_0)_x, \quad H'_1(u, v) = (u_0 v_1 + u_1 v_0)_x, \quad H'_2(u, v) = (u_0 v_2 + u_1 v_1 + u_2 v_0)_x.$$

Comparing the coefficients of like powers of  $p$ , we obtain

$$u_0(x, t) = \sin x, \quad v_0(x, t) = \sin x;$$

$$u_1(x, t) = -\sin x \frac{t^a}{\Gamma(1+a)}, \quad v_1(x, t) = -\sin x \frac{t^b}{\Gamma(1+b)};$$

$$u_2(x,t) = (\sin x - \sin 2x) \frac{t^{2a}}{\Gamma(1+2a)} + \sin 2x \frac{t^{a+b}}{\Gamma(1+a+b)},$$

$$v_2(x,t) = (\sin x - \sin 2x) \frac{t^{2b}}{\Gamma(1+2b)} + \sin 2x \frac{t^{a+b}}{\Gamma(1+a+b)};$$

$$u_3(x,t) = (-\sin x + 5\sin 2x + 3\cos 2x \sin x) \frac{t^{3a}}{\Gamma(1+3a)} + \sin 2x \frac{\Gamma(1+2a)}{\Gamma(1+a)^2} \frac{t^{3a}}{\Gamma(1+3a)} -$$

$$-(4 + \frac{\Gamma(1+a+b)}{\Gamma(1+a)\Gamma(1+b)}) \sin 2x \frac{t^{2a+b}}{\Gamma(1+2a+b)} - (\sin 2x - \sin x - 3\cos 2x \sin x) \frac{t^{a+2b}}{\Gamma(1+a+2b)}$$

$$v_3(x,t) = (-\sin x + 5\sin 2x - 3\cos 2x \sin x) \frac{t^{3b}}{\Gamma(1+3b)} + \sin 2x \frac{\Gamma(1+2b)}{\Gamma(1+b)^2} \frac{t^{3b}}{\Gamma(1+3b)} -$$

$$-(4 + \frac{\Gamma(1+a+b)}{\Gamma(1+a)\Gamma(1+b)}) \sin 2x \frac{t^{a+2b}}{\Gamma(1+a+2b)} - (\sin 2x - \sin x - 3\cos 2x \sin x) \frac{t^{2a+b}}{\Gamma(1+2a+b)}$$

Proceeding in the similar manner, the rest of the components of  $u_n(x, t)$  and  $v_n(x, t)$ ,  $n \geq 0$ , can be obtained and the series solutions are thus entirely determined. Finally, we approximate the analytical solution of  $u(x, t)$  and  $v(x, t)$  by the truncated series as given in equation (15).

Now for the standard case i.e. for  $a=b=1$ ,  $u(x, t)$  and  $v(x, t)$ ,  $u(x, t)=v(x, t)=e^{-t} \sin x$ , the result is the same as the result of Abdou and Soliman [18].

**Example 2.**

We will solve the nonlinear time fractional coupled KdV equations

$$D_t^a u + u_{xxx} + 2uu_x + 2u_x v = 0 \tag{22}$$

$$D_t^b v + v_{xxx} + 2vv_x + uv_x = 0 \tag{23}$$

where  $0 < x \leq 1$  and  $0 < t \leq 1$   
with the initial conditions as

$$u(x,0) = e^{-kx} \tag{24}$$

$$v(x,0) = -e^{-kx} \tag{25}$$

Taking the Laplace transform of equation (22) and (23),

$$L[D_t^a u] + L[u_{xxx}] + 2L[uu_x] + 2L[u_x v] = 0$$

$$L[D_t^b v] + L[v_{xxx}] + 2L[vv_x] + L[uv_x] = 0$$

Using equation (4) and then applying the inverse Laplace transform, we obtain

$$u(x,t) = L^{-1} \left[ \frac{1}{s} u(x,0) - \frac{1}{s^a} \{L[u_{xxx}] - 2L[uu_x] - 2L[u_x v]\} \right] \tag{26}$$

$$v(x,t) = L^{-1} \left[ \frac{1}{s} v(x,0) - \frac{1}{s^b} \{L[v_{xxx}] - 2L[vv_x] - L[uv_x]\} \right] \tag{27}$$

To apply the homotopy perturbation method to equations (26) and (27), we express

$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t)$  and  $v(x,t) = \sum_{n=0}^{\infty} p^n v_n(x,t)$ . The nonlinear terms can be decomposed as equation (8).

Equations (26) and (27) reduce to

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = u_0(x,t) + L^{-1} \left[ -\frac{1}{s^a} \{L[(\sum_{n=0}^{\infty} p^n u_n)_{xxx}] + 2L[\sum_{n=0}^{\infty} p^n H_n(u)] + 2L[\sum_{n=0}^{\infty} p^n H'_n(u,v)]\} \right]$$

$$\sum_{n=0}^{\infty} p^n v_n(x,t) = v_0(x,t) + L^{-1} \left[ -\frac{1}{s^b} \{L[(\sum_{n=0}^{\infty} p^n v_n)_{xxx}] + 2L[\sum_{n=0}^{\infty} p^n G_n(v)] + 2L[\sum_{n=0}^{\infty} p^n G'_n(u,v)]\} \right]$$

Where  $H_n(u)$ ,  $G_n(v)$ ,  $H'_n(u,v)$  and  $G'_n(u,v)$  are He's polynomials that represent nonlinear terms  $uu_x$ ,  $vv_x$ ,  $u_xu$  and  $uv_x$  respectively, which are given as

$$H_0(u) = u_0(u_0)_x, \quad H_1(u) = u_0(u_1)_x + u_1(u_0)_x, \quad H_2(u) = u_0(u_2)_x + u_1(u_1)_x + u_2(u_0)_x,$$

$$G_0(v) = v_0(v_0)_x, \quad G_1(v) = v_0(v_1)_x + v_1(v_0)_x, \quad G_2(v) = v_0(v_2)_x + v_1(v_1)_x + v_2(v_0)_x,$$

$$H'_0(u,v) = (u_0)_x v_0, \quad H'_1(u,v) = (u_0)_x v_1 + (u_1)_x v_0, \quad H'_2(u,v) = (u_0)_x v_2 + (u_1)_x v_1 + (u_2)_x v_0,$$

$$G'_0(u,v) = (v_0)_x u_0, \quad G'_1(u,v) = (v_0)_x u_1 + (v_1)_x u_0, \quad G'_2(u,v) = (v_0)_x u_2 + (v_1)_x u_1 + (v_2)_x u_0.$$

Comparing the coefficients of like powers of  $p$ , we obtain

$$u_0(x,t) = e^{-kx}, \quad v_0(x,t) = -e^{-kx}; \quad u_1(x,t) = k^3 e^{-kx} \frac{t^a}{\Gamma(1+a)}, \quad v_1(x,t) = -k^3 e^{-kx} \frac{t^b}{\Gamma(1+b)};$$

$$u_2(x,t) = k^4 e^{-2kx} \left[ (2+k^2 e^{kx}) \frac{t^{2a}}{\Gamma(1+2a)} - 2 \frac{t^{a+b}}{\Gamma(1+a+b)} \right]$$

$$v_2(x,t) = k^4 e^{-2kx} \left[ (2-k^2 e^{kx}) \frac{t^{2a}}{\Gamma(1+2a)} - 2 \frac{t^{a+b}}{\Gamma(1+a+b)} \right]$$

$$u_3(x,t) = k^5 e^{-3kx} \left[ (4+18k^2 e^{kx} + k^4 e^{2kx}) \frac{t^{3a}}{\Gamma(1+3a)} + 2k^2 e^{kx} \frac{\Gamma(1+2a)}{\Gamma(1+a)^2} \frac{t^{3a}}{\Gamma(1+3a)} \right. \\ \left. (-4-8k^2 e^{kx} - k^2 e^{kx} \frac{\Gamma(1+a+b)}{\Gamma(1+a)\Gamma(1+b)}) \frac{t^{2a+b}}{\Gamma(1+2a+b)} + 2(2-k^2 e^{kx}) \frac{t^{a+2b}}{\Gamma(1+a+2b)} \right]$$

$$v_3(x,t) = -k^5 e^{-3kx} \left[ (4-18k^2 e^{kx} + k^4 e^{2kx}) \frac{t^{3b}}{\Gamma(1+3b)} - 2k^2 e^{kx} \frac{\Gamma(1+2b)}{\Gamma(1+b)^2} \frac{t^{3b}}{\Gamma(1+3b)} - \right. \\ \left. -2(-4-8k^2 e^{kx} - k^2 e^{kx} \frac{\Gamma(1+a+b)}{\Gamma(1+a)\Gamma(1+b)}) \frac{t^{a+2b}}{\Gamma(1+a+2b)} + 2(2+k^2 e^{kx}) \frac{t^{2a+b}}{\Gamma(1+2a+b)} \right]$$

Proceeding in the similar manner, the rest of the components of  $u_n(x, t)$  and  $v_n(x, t)$ ,  $n \geq 0$  can be obtained and the series solutions are thus entirely determined. Finally, we approximate the analytical solution of  $u(x, t)$  and  $v(x, t)$  by the truncated series as given in equation (15).

#### 4. Numerical results and discussion

In this section, the numerical results of the probability density functions  $u(x, t)$  and  $v(x, t)$  are calculated for different fractional Brownian motions  $a=b=0.80, 0.90$  and also for the standard motion  $a=b=1$  for different particular cases and the results are depicted in Figs. 1-4. During numerical computations only four iterations are considered. It is evident that by using more terms, the accuracies of the results can be improved.

The fig.1 depicts that for  $b=1$  the probability density function  $u(x, t)$  initially decreases with the increase both in  $t$  and  $a$  and afterwards it becomes opposite in nature. Fig.3 demonstrates the variations of the probability density function  $v(x, t)$  with  $t$  and  $b$  for  $a=1$ . It is clear that the nature of the behavior of  $v(x, t)$  is identical to  $u(x, t)$  described in fig.1

Figs.2 and 4 are the 3-D representations of  $u(x, t)$  and  $v(x, t)$  with  $t$  and  $x$  at  $a=b=1/2$ . For the standard motion i.e., at  $a=b=1$ , Figs. 5 and 6 respectively depict the comparison of numerical values of  $u(x, t)$  and  $v(x, t)$  with the exact results obtained by [17, 18].

Figs.7 and 8 show the numerical analysis of the absolute error between the exact solution and approximate solution obtained by HPTM in the regions  $0 < x \leq 1$  and  $0 < t \leq 1$  when  $a=b=1$ .

The fig.9 depicts that for  $b=1$  the probability density function  $u(x, t)$  increases with the increase in  $t$  and decreases with increase in  $a$ .

The fig. 11 depicts that for  $a=1$  the probability density function  $v(x, t)$ . Figs.10 and 12 are the 3-D representations of  $u(x, t)$  and  $v(x, t)$  with  $t$  and  $x$  at  $a=1/2, b=1$  and  $a=1, b=1/2$  respectively.

Figs.13 and 14 respectively show the comparisons of numerical values of  $u(x, t)$  and  $v(x, t)$  with the exact results which are as follows [24]  $U(x, t) = C_1 \exp(-k[x - (k^2 + 2C_2)t])$  and  $V(x, t) = C_1 \exp(-k[x - (k^2 + 2C_2)t]) + C_2$ , where  $C_1, C_2$  are the integration constants. When  $C_1=1, C_2=0$  and  $k=1$ .

It is clear from obtained results that in short time interval the results obtained by HPTM with only four iterations are almost accurate [24].

### 5. Conclusion

In the present article the authors have succeed to achieve two important goals. First one is the study of fundamental aspect of fractional calculus which clearly exhibits the concept of Non-Markovian nature of the fractional order systems which is an extension of the classical standard order mathematical models. Thus fractional order system captures the phenomena of memory effects which are simply ignored by the integer order system. The second goal of the study is the successful drive towards graphical presentation of the approximate solution for different fractional Brownian motions for different particular cases of the coupled Burgers equations and coupled KdV equations which have immense applications in physics and engineering.

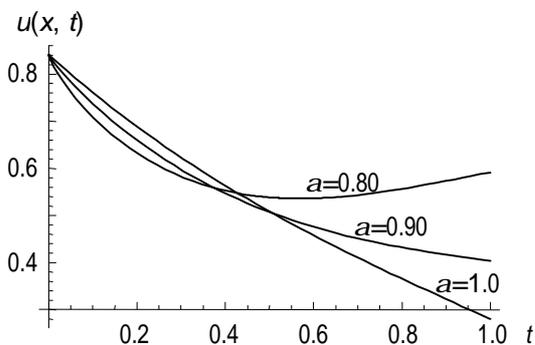


Fig.1. Plot of  $u(x, t)$  vs.  $t$  for different values of  $a$  and  $b=1$

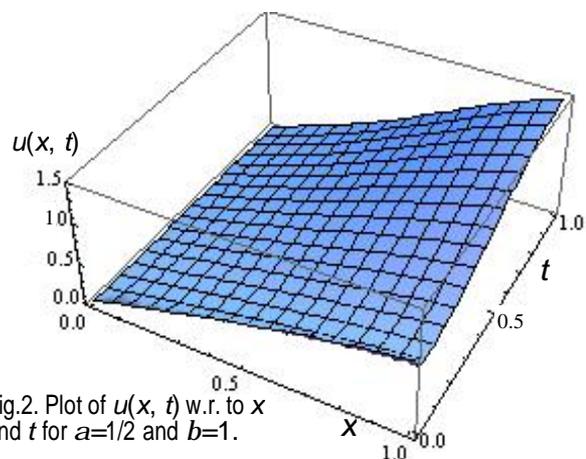


Fig.2. Plot of  $u(x, t)$  w.r. to  $x$  and  $t$  for  $a=1/2$  and  $b=1$ .

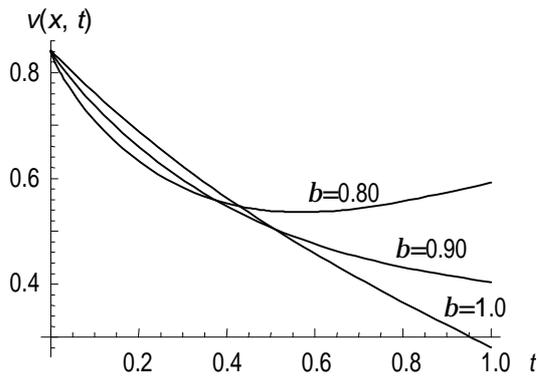


Fig.3. Plot of  $v(x, t)$  vs.  $t$  for different values of  $b$  and  $a=1$

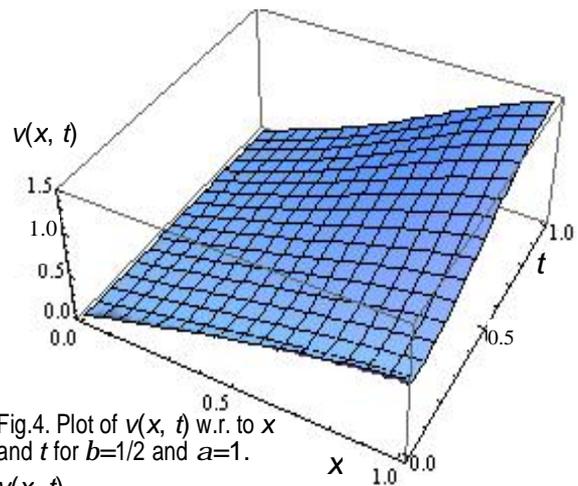


Fig.4. Plot of  $v(x, t)$  w.r. to  $x$  and  $t$  for  $b=1/2$  and  $a=1$ .

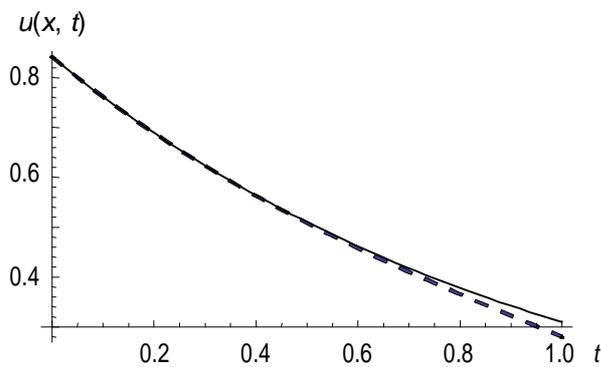


Fig.5. Comparison of  $u(x, t)$  with exact solution at  $a=1$  w.r. to  $t$

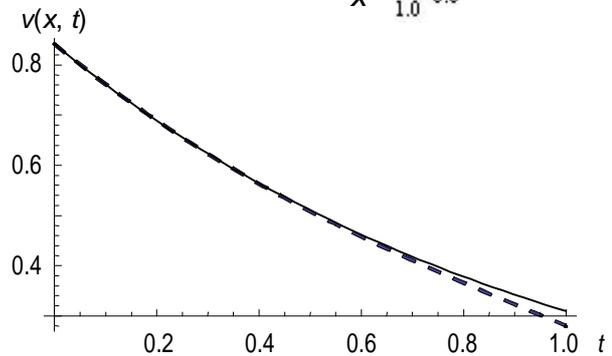


Fig.6. Comparison of  $v(x, t)$  with exact solution at  $a=1$  w.r. to  $t$

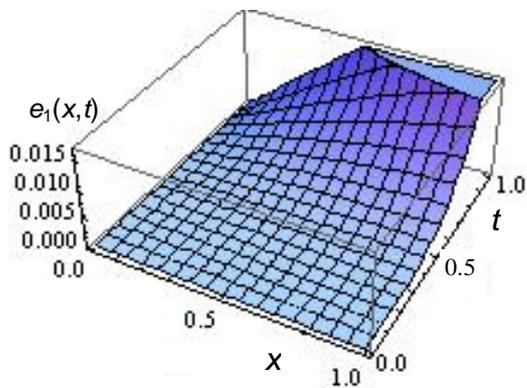


Fig.7. Error  $e_1(x, t) = |u_{exact}(x, t) - u_{approx}(x, t)|$

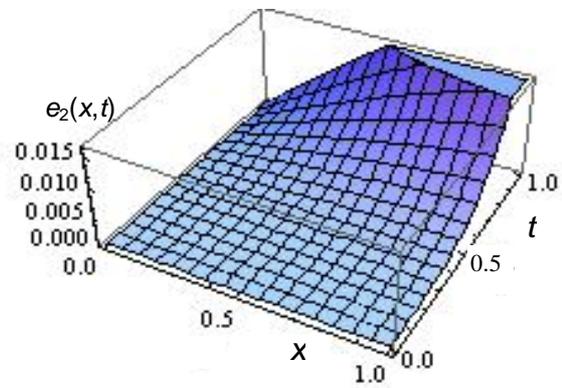


Fig.8. Error  $e_2(x, t) = |v_{exact}(x, t) - v_{approx}(x, t)|$

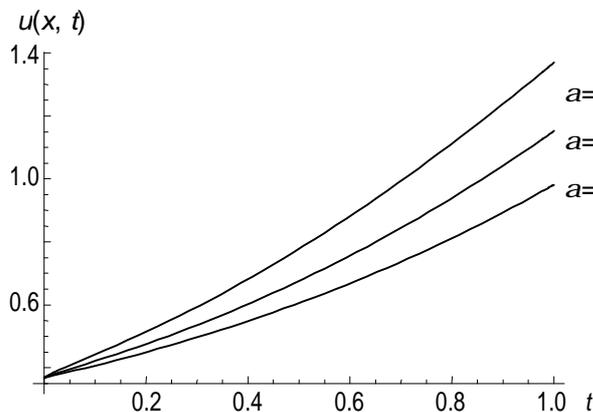


Fig.9. Plot of  $u(x, t)$  vs.  $t$  for different values of  $a$  and  $b=1$

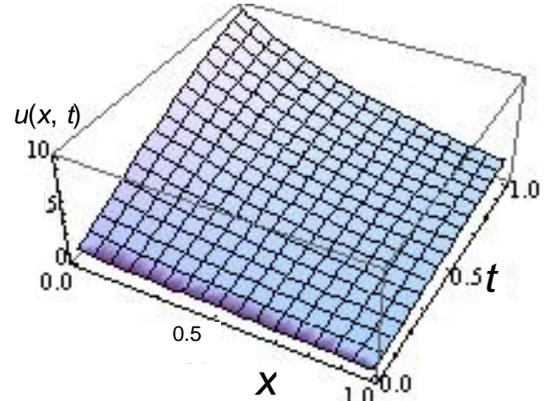


Fig.10. Plot of  $u(x, t)$  w.r. to  $x$  and  $t$  for  $a=1/2$ ,  $b=1$

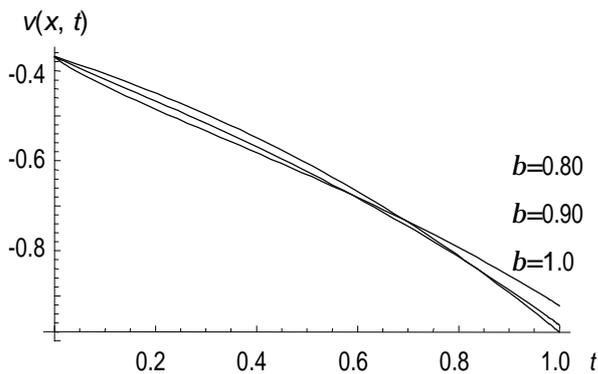


Fig.11. Plot of  $v(x, t)$  vs.  $t$  for different values of  $b$  and  $a=1$

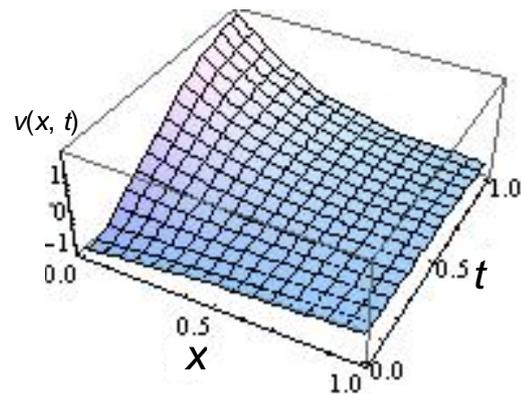


Fig.12. Plot of  $v(x, t)$  w.r. to  $x$  and  $t$  for  $b=1/2$ ,  $a=1$

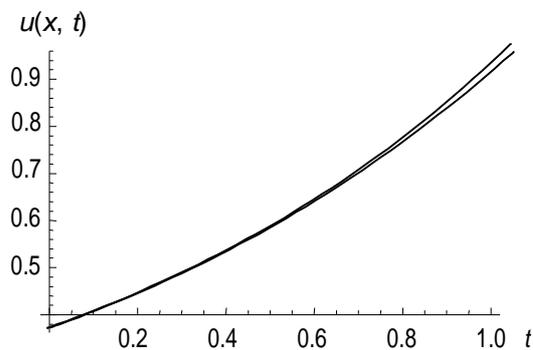


Fig.13. Comparison of  $u(x, t)$  with exact solution at  $a=1$  and  $b=1$  w.r. to  $t$ .

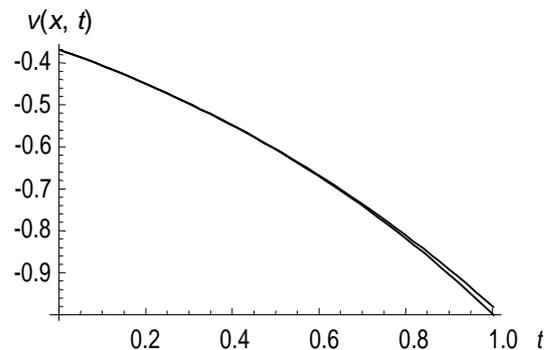


Fig.14. Comparison of  $v(x, t)$  with exact solution at  $a=1$  and  $b=1$  w.r. to  $t$

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## The pfaffian technique: a (2+1)-dimensional Korteweg - de Vries equation

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The (2+1)-dimensional Korteweg-de Vries (KdV) equation, which was first derived by Boiti et al (Boiti M., Leon J.J.P., Manna M. and Pempinelli F. in 1986), has been studied by various distinct methods. It is known that this (2+1)-dimensional KdV equation has rich solutions, such as multi-soliton solutions, dromion solutions. In the present article, a unified representation of its  $N$ -soliton solution is given by means of pfaffians. We will show that this (2+1)-dimensional KdV equation is nothing but the Plücker identity when its  $t$ -function is presented by pfaffian.

### 1. Introduction

The solitary wave, so-called because it often occurs as a single entity and is localized, was first observed by J.Scott Russell on the Edinburgh-Glasgow Canal in 1834. Now, it is known that many nonlinear evolution equations have soliton solutions, such as the Korteweg-de Vries equation, the Sin-Gordon equation, the nonlinear Schrödinger equation, the Kadomtsev-Petviashvili (KP) equation, the Davey-Stewartson (DS) equation, and so on. Many methods have been developed to derive solitary wave solution or soliton solution to nonlinear evolution equations. Some of the most important methods are the inverse scattering transformation (IST) [1], the bilinear method [2-7], symmetry reductions [8], Bäcklund and Darboux transformations [9] and so on. Possessing soliton solutions is viewed as one of the basic integrable properties of nonlinear evolution equations.

In this paper, we are interested in the general expression of  $N$ -soliton solution to the (2+1)-dimensional KdV equation,

$$\frac{\partial u}{\partial t} + 3\frac{\partial(vu)}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 \tag{1}$$

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0 \tag{2}$$

which was first derived by Boiti et al. [10] using the idea of the weak Lax pair. The equation system (1) and (2) can also be obtained from the inner parameter-dependent symmetry constraint of the KP equation [11]. In [12], the dromion structure of the (2+1)-dimensional KdV equation was studied by Lou etc., besides, some exact solutions to it were also discussed by Wazwaz [13].

To find general expression of multi-soliton solutions for nonlinear evolution equations in (2+1) dimensions is much more difficult than in (1+1) dimensions. To a certain point, for soliton equations having 3-soliton solutions, pfaffian technique provides us an approach to determine whether soliton equations have multi-soliton solutions or not. In this article, we present  $N$ -soliton solution to the (2+1)-dimensional KdV equation (1) and (2) in a compact way and give a strict proof by means of pfaffian identities.

We first review some properties of pfaffians. Pfaffians are antisymmetric functions with respect to its independent variables  $pf(a, b) = -pf(b, a)$ , for any  $a$  and  $b$ .

A  $2n$ -th degree pfaffian is inductively defined by the expansion rule [3]

$$pf(1, 2, \dots, 2n) = \sum_{j=2}^{2n} (-1)^j pf(1, j)(2, 3, \dots, \hat{j}\sqrt{b^2 - 4ac}, \dots, 2n),$$

Where  $\hat{j}$  denotes the absence of  $j$ . For example, when  $n=2$ , we have

$$pf(1,2,3,4) = pf(1,2)pf(3,4) - pf(1,3)pf(2,4) + pf(1,4)pf(2,3)$$

There are various kinds of pfaffian identities. In this article, we mainly use the so called Plücker relation for pfaffians [3]

$$pf(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, 1, 2, \dots, 2n-1)pf(1, 2, \dots, 2n) = pf(\mathbf{a}_1, 1, 2, \dots, 2n-1)pf(\mathbf{a}_2, \mathbf{a}_3, 1, 2, \dots, 2n) - pf(\mathbf{a}_2, 1, 2, \dots, 2n-1)pf(\mathbf{a}_1, \mathbf{a}_3, 1, 2, \dots, 2n) + pf(\mathbf{a}_3, 1, 2, \dots, 2n-1)pf(\mathbf{a}_1, \mathbf{a}_2, 1, 2, \dots, 2n) \quad (3)$$

Hereafter, for simplicity, we denote  $pf(1, 2, \dots, 2n) = (1, 2, \dots, 2n)$ .

## 2. N-soliton solution to the (2+1) dimensional KdV equation

By using the dependent variable transformations

$$u = 2 \frac{\partial^2 \ln f(x, y, t)}{\partial x \partial y}, \quad v = 2 \frac{\partial^2 \ln f(x, y, t)}{\partial x^2} \quad (4)$$

the (2+1)-dimensional KdV equation system (1) and (2) can be rewritten as

$$D_y[D_t + D_x^3]f \cdot f = 0 \quad (5)$$

where the Hirota bilinear operators  $D$ 's are defined by

$$D_x^m D_t^n a \cdot b \equiv \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^m \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^n a(x, t) b(x', t') \Big|_{x'=x, t'=t}$$

with  $n$  and  $m$  are arbitrary nonnegative integers. 3-soliton solution to (2+1)-dimensional KdV equation (5) is derived by using perturbation method: [13]

$$f = 1 + \sum_{i=1}^3 \exp(\mathbf{h}_i) + \sum_{1 \leq i < j \leq 3} c_{ij} \exp(\mathbf{h}_i + \mathbf{h}_j) + c_{123} \exp(\mathbf{h}_1 + \mathbf{h}_2 + \mathbf{h}_3) \quad (6)$$

where

$$\begin{aligned} \mathbf{h}_i &= k_i x + m_i y + q_i t + \mathbf{h}_i^0, & q_i &= -k_i^3, & \mathbf{h}_i^0 & \text{(for } i=1, 2, 3) \text{ are arbitrary constants} \\ c_{ij} &= \frac{(p_i - p_j)(m_i - m_j)}{(p_i + p_j)(m_i + m_j)}, & & & & \text{for } i, j = 1, 2, 3 \end{aligned}$$

Although we can obtain its multi-soliton solutions theoretically by using perturbation method recursively, it seems impractical to deduce an explicit expression of the  $N$ -soliton solution. In this Letter, we express the  $N$ -soliton solution to the (2+1)-dimensional KdV equation in terms of pfaffians in a compact way and give a strict proof.

The expression of 3-soliton solution (6) suggests that  $N$ -soliton solutions to (2+1)-dimensional (5) are expressed by pfaffians. In fact, we have

$$f = (d_0, \mathbf{b}_0, a_1, a_2, \dots, a_N, b_1, b_2, \dots, b_N) \quad (7)$$

where the entries of the pfaffian are defined as follows:

$$\begin{aligned} (d_0, a_j) &= \exp(\mathbf{h}_j), & (d_0, b_j) &= -1, & (d_0, \mathbf{b}_0) &= 1, \\ (a_j, a_k) &= a_{jk} \exp(\mathbf{h}_j + \mathbf{h}_k), & (a_j, b_k) &= d_{jk}, \\ (b_j, \mathbf{b}_0) &= 1, & (a_j, \mathbf{b}_0) &= 0, & (b_j, b_k) &= b_{jk}, \\ & & \text{with } \mathbf{h}_j &= p_j x + m_j y - p_j^3 t + \mathbf{h}_j^0, \end{aligned}$$

$$a_{jk} = \frac{p_j - p_k}{p_j + p_k}, \quad b_{jk} = \frac{m_j - m_k}{m_j + m_k}, \quad d_{jk} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

for  $j, k = 1, 2, \dots, N$ .

In the following, we will prove that pfaffian function  $f$  given by (7) satisfies (2+1)-dimensional KdV equation (5). Using the method described in [15], we obtain the following differential formulae

$$\frac{\partial f}{\partial x} = -(d_0, d_1, a_1 \dots a_N, b_1 \dots b_N), \quad \frac{\partial^2 f}{\partial x^2} = -(d_0, d_2, a_1 \dots a_N, b_1 \dots b_N) \quad (8)$$

$$\frac{\partial^3 f}{\partial x^3} = -(d_0, d_3, a_1 \dots a_N, b_1 \dots b_N) - (d_0, d_1, d_2, b_0, a_1 \dots a_N, b_1 \dots b_N) \quad (9)$$

$$\frac{\partial f}{\partial t} = (d_0, d_3, a_1 \dots a_N, b_1 \dots b_N) - 2(d_0, d_1, d_2, b_0, a_1 \dots a_N, b_1 \dots b_N) \quad (10)$$

Where  $d_n$  are defined by as follows:

$$(d_n, b_j) = (d_n, d_m) = (d_n, b_0) = (d_n, d_0) = 0 \quad (11)$$

$$(d_n, a_j) = p_j^n \exp(h_j), \quad (n, m = 1, 2, \dots, j = 1, 2, \dots, N) \quad (12)$$

Furthermore, in order to prove  $N$ -soliton solution  $f$ , we have to introduce a new character  $b_3$  defined by [14]:

$$(b_3, a_j) = 0, \quad (b_3, b_j) = -m_j, \quad (b_3, b_0) = 0, \quad (b_3, d_n) = 0 \quad (13)$$

$$\text{for } j = 1, 2, \dots, N, \quad n = 1, 2, 3, \dots \quad (14)$$

With the assistance of  $b_3$ , we have

$$\frac{\partial f}{\partial y} = (d_0, b_3, a_1 \dots a_N, b_1 \dots b_N) \quad (15)$$

$$\frac{\partial^2 f}{\partial x \partial y} = (d_0, d_1, b_3, b_0, a_1 \dots a_N, b_1 \dots b_N) \quad (16)$$

$$\frac{\partial^3 f}{\partial x^2 \partial y} = (d_0, d_2, b_3, b_0, a_1 \dots a_N, b_1 \dots b_N) \quad (17)$$

$$\frac{\partial^4 f}{\partial x^3 \partial y} = (d_0, d_3, b_3, b_0, a_1 \dots a_N, b_1 \dots b_N) - (d_0, d_1, d_2, b_3, a_1 \dots a_N, b_1 \dots b_N) \quad (18)$$

$$\frac{\partial^2 f}{\partial y \partial t} = -(d_0, d_3, b_3, b_0, a_1 \dots a_N, b_1 \dots b_N) - 2(d_0, d_1, d_2, b_3, a_1 \dots a_N, b_1 \dots b_N) \quad (19)$$

Denoting  $(d_0, a_1, \dots, a_N, b_1, \dots, b_N) = (\dots)$ ,  $(d_0, d_2, b_3, b_0, a_1, \dots, a_N, b_1, \dots, b_N) = -(d_2, b_3, b_0, \dots)$  using these relations, Eq. (5) is reduced to

$$\begin{aligned} D_y [D_t + D_x^3] f \cdot f &= f \left( \frac{\partial^2 f}{\partial y \partial t} + \frac{\partial^4 \Omega}{\partial x^3 \partial y} \right) - 3 \frac{\partial f}{\partial x} \frac{\partial^3 f}{\partial x^2 \partial y} + 3 \frac{\partial^2 f}{\partial x \partial y} \frac{\partial^2 f}{\partial x^2} - \frac{\partial f}{\partial y} \left( \frac{\partial f}{\partial t} + \frac{\partial^3 f}{\partial x^3} \right) = \\ &= 3[(d_1, \dots)(d_2, b_3, b_0, \dots) - (d_2, \dots)(d_1, b_3, b_0, \dots) + (b_3, \dots)(d_1, d_2, b_0, \dots) - (b_0, \dots)(d_1, d_2, b_3, \dots)] = 0. \end{aligned}$$

So far, we have finished proving  $N$ -soliton solution  $f$  (7) to the (2+1)-dimensional KdV (5). The  $N$ -soliton solution to nonlinear system of equations (1) and (2) are given by

$$u = 2 \frac{\partial^2 \ln f(x, y, t)}{\partial x \partial y}, \quad v = 2 \frac{\partial^2 \ln f(x, y, t)}{\partial x^2}$$

$$f = (d_0, \mathbf{b}_0, a_1, a_2, \dots, a_N, b_1, b_2, \dots, b_N)$$

### 3. Conclusion

To sum up, the multi-soliton solution to the (2+1)-dimensional KdV equation is derived and proved by pfaffian identity. As we can see from the procedure, the key point is to derive suitable expressions for differential formulae of  $f$ . The method used in this article has been applied to a vector potential KdV equation and Ito equation [14], a coupled Ramani equation [15] and a higher order Ito equation [16], etc. It is worth noting that this method is valid for many other equations to determine their multi-soliton solution. Besides, many new properties of pfaffians are discovered through studying soliton solution to nonlinear evolution equations. A natural question is whether there are other types of solutions that can be expressed by pfaffians, for example the solutions to supersymmetric equations. We expect that more applications of pfaffians are to be found out.

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# A study on properties of curvature, Carathéodory and Kobayashi metrics

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Some basic examples, important properties, and some applications of the curvature of conformal metric will be discussed in this paper. Two metrics called Carathéodory metric and Kobayashi metric will be studied in detail discussing their invariant properties and applications.

## 1. Introduction

The study of curvature of conformal metrics has been instrumental in providing new proofs of classical results in complex analysis. The curvature also helps us to describe the qualitative difference between conformal metrics. Named after the Greek mathematician Constantin Carathéodory, the Carathéodory metric is constructed by using the greatest possible derivative at a point of maps from a domain  $\Omega$  in  $\mathbb{C}$  into the unit disc. A dual construction results in a metric called the Kobayashi metric. These two metrics are invariant under conformal mappings and are motivated by the extremal problem that arises from the Riemann Mapping Theorem.

In section two of this paper, we will define curvature of a conformal metric on a domain  $\Omega$  in  $\mathbb{C}$  followed by discussing curvature of some known metrics. Some important properties and applications of curvature to prove important results in complex analysis will be explained in detail in the form of Lemmas and Theorems. Section three discusses several examples, properties, and applications of two different types of metrics, called Carathéodory metric and Kobayashi metric. In section four, we discuss the completeness of these two metrics on a reasonable class of domains. The last section is devoted in explaining some concepts related to curvature and nondegeneracy of the Kobayashi metric and the hyperbolicity of a domain. Although this paper does not contain any new results, it has been designed to present the proofs of some well-known theorems in complex analysis in a systematical manner. Many results are directly cited from [6] in the main text. No other referencing have been included but they are provided at the end of the paper.

## 2. Curvature

This section is devoted to the definition, examples, properties, and applications of curvature of a metric at a point.

### Definition 2.1 (Curvature).

Let  $r$  be a conformal metric on a domain  $\Omega \subseteq \mathbb{C}$  and define

$$S = S(r) = \{z \in \Omega : r(z) \neq 0, \text{ and } r \text{ is } c^2 \text{ near } z\}$$

The curvature,  $k_r$  of the metric  $r$  at a point  $z \in S$  is the function from  $S$  to  $\mathbb{R}$  defined by

$$k_r(z) = \frac{-\Delta \log(r(z))}{(r(z))^2}, \quad \text{where } \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4 \frac{\partial^2}{\partial z \partial \bar{z}} \quad (1)$$

### 2.1. Examples.

In this subsection, we calculate the curvature of some known conformal metrics.

#### Example 2.2 (curvature of the Euclidean metric).

If  $r(z)=1$  (i.e.  $r$  is the Euclidean metric on  $\Omega$ ) then  $k_r(z)=0$ .

This is true because  $\log(1)=0$ . Since the Euclidean metric does not change from point to point, the curvature  $k_r(z)=0$  makes sense.

Example 2.3 (curvature of Poincaré metric).

Let the unit disc,  $\mathbf{D}$  be equipped with the Poincaré metric  $r(z) = \frac{1}{1-|z|^2}$ ,  $z \in \mathbf{D}$ . Then,

$$k_r(z) = -4, \quad \forall z \in \mathbf{D}.$$

To justify this, we first calculate  $-\Delta \log[r(z)]$ .

$$-\Delta \log(r(z)) = -\Delta \log\left(\frac{1}{1-|z|^2}\right) = \Delta \log(1-|z|^2)$$

But,  $|z|^2 = z \cdot \bar{z}$  and  $\Delta = 4 \frac{\partial^2}{\partial z \partial \bar{z}}$ , so

$$\Delta \log(1-|z|^2) = \Delta \log(1-z\bar{z}) = 4 \frac{\partial^2}{\partial z \partial \bar{z}} (\log(1-z\bar{z})) = 4 \frac{\partial}{\partial z} \left( -\frac{z}{1-z\bar{z}} \right) = -\frac{4}{(1-|z|^2)^2} \quad (2)$$

The curvature of Poincaré metric  $r$  is

$$k_r(z) = \frac{-\Delta \log(r(z))}{(r(z))^2} = \frac{-\frac{4}{(1-|z|^2)^2}}{\left(\frac{1}{1-|z|^2}\right)^2} = -4$$

Therefore, the Poincaré metric on  $\mathbf{D}$  has constant negative curvature  $-4$ .

Example 2.4 (curvature of spherical metric).

Consider a spherical metric  $s(z) = \frac{2}{1+|z|^2}$  on  $\mathbf{C}$ . Then  $k_s(z) \equiv 1, \forall z \in \mathbf{C}$ .

This is justified immediately using similar steps of the previous example.

$$k_s(z) = -\frac{\Delta \log(s(z))}{s(z)^2} = \frac{\frac{4}{(1+|z|^2)^2}}{\frac{4}{(1+|z|^2)^2}} = 1 \quad (3)$$

Example 2.5 (curvature of a metric on a disc).

Let  $D(0, r)$  be the open disc of radius  $r$  and center  $0$ . We define the conformal metric  $r_r$  on  $D(0, r)$  by  $r_r(z) = r/(r^2 - |z|^2)$ .

Then  $r_r$  has a constant curvature of  $-4$  where  $r_r$  is the analogue of the Poincaré metric for  $D(0, r)$ . The proof is similar to example 2.3.

**2.2. Properties.**

Before we discuss some important properties of curvature, we first prove the following lemma.

Lemma 2.6.

If  $f$  is a non-vanishing holomorphic function in  $\Omega$ , then  $\log(|f|)$  is harmonic.

Proof: We wish to show  $\Delta \log(|f|) = 0$ . But,

$$\Delta \log(|f|) = \Delta \log\left((f \cdot \bar{f})^{\frac{1}{2}}\right) = \frac{1}{2} \Delta (\log(f) + \log(\bar{f})) = 2 \frac{\partial}{\partial z} \left( \frac{1}{f} \cdot \frac{\partial f}{\partial z} \right) + 2 \frac{\partial}{\partial z} \left( \frac{\partial}{\partial \bar{z}} (\log(\bar{f})) \right)$$

Since  $f$  is holomorphic,  $\partial f / \partial \bar{z} = 0$  and hence the first term in the right side vanishes. The second term also vanishes because

$$\frac{\partial}{\partial z} \left( \frac{\partial}{\partial \bar{z}} (\log(\bar{f})) \right) = \frac{\partial}{\partial \bar{z}} \left( \frac{\partial}{\partial z} (\log(\bar{f})) \right) = \frac{\partial}{\partial \bar{z}} \left( \frac{1}{\bar{f}} \cdot \frac{\partial \bar{f}}{\partial z} \right) = \frac{\partial}{\partial \bar{z}} \left( \frac{1}{\bar{f}} \cdot \frac{\partial \bar{f}}{\partial \bar{z}} \right) = 0 \quad (4)$$

Similar reasoning implies that  $\log(|f|^2)$  is also harmonic if  $f$  is non-vanishing and holomorphic on  $\Omega$ .

**Proposition 2.7 (conformal invariant).**

Let  $\Omega_1$  and  $\Omega_2$  be planar domains and  $f: \Omega_1 \rightarrow \Omega_2$  a conformal map (i.e.  $f \neq 0$ ). If  $r$  is a metric on  $\Omega_2$  then,

$$k_{f^{\hat{a}}r}(z) = k_r(f(z)) \quad \forall z \in f^{-1}(S(r)).$$

*Proof:*

We know that the pullback of  $r$  under  $f$  i.e.  $f^{\hat{a}}r$  is a metric on  $\Omega_1$ . We will show that the curvature of  $f^{\hat{a}}r$  at  $z$  is equal to the curvature of  $r$  at  $f(z)$ . Now,

$$\begin{aligned} k_{f^{\hat{a}}r}(z) &= \frac{-\Delta \log(f^{\hat{a}}r(z))}{(f^{\hat{a}}r(z))^2} = \frac{-\Delta \log(r(f(z)) \cdot |f'(z)|)}{(r(f(z)) \cdot |f'(z)|)^2} = \\ &= \frac{-\Delta \log(r(f(z))) - \Delta \log(|f'(z)|)}{(r(f(z)) \cdot |f'(z)|)^2} = \frac{-\Delta \log(r(f(z)))}{(r(f(z)) \cdot |f'(z)|)^2} \end{aligned}$$

since  $f \neq 0$  is holomorphic and non-vanishing,  $\log(|f'(z)|)$  is harmonic using Lemma 2.1. But,

$$-\Delta \log(r(f(z))) = \Delta \log r(f(z)) \cdot |f'(z)|^2.$$

Hence

$$k_{f^{\hat{a}}r}(z) = \frac{\Delta \log r(f(z))}{(r(f(z)))^2} = k_r(f(z)).$$

**2.3. Applications.**

In this subsection, we explain some of the applications of the curvature of conformal metric to prove important results in complex analysis. We start with a version of Schwarz Lemma called ‘‘Ahlfors’s version of Schwarz’s Lemma’’ about curvature.

**Theorem 2.8 (Ahlfors's Version of Schwarz Lemma about curvature).**

Let the domain  $\Omega \subseteq \mathbb{C}$  be equipped with a metric  $S$  that has curvature not exceeding  $-4$  at all points of  $\Omega$ . Let  $D$  be equipped with the Poincaré metric  $r$ . If  $f: D \rightarrow \Omega$  is holomorphic then,

$$f^{\hat{a}}S(z) \leq r(z), \quad \forall z \in D$$

*Proof:*

Let  $0 < r < 1$ . Define a conformal metric  $r_r(z) = \frac{r}{r^2 - |z|^2}$  on  $D(0, r)$  as in example 2.5. Then

$r_r$  has a constant curvature  $-4$ . Define  $g = f^{\hat{a}}S / r_r$ .

As  $g$  is continuous, non-negative on  $D(0, r)$  and that  $g \rightarrow 0$  when  $|z| \rightarrow r$ ,  $g$  attains a maximum value  $M$  at some point  $x \in D(0, r)$ .

*Claim:*  $M \leq 1$ .

*Proof of the Claim:* If  $f^{\hat{a}}S(x) = 0$  then there is nothing to prove. Assume that  $f^{\hat{a}}S(x) > 0$ . Since  $k_{S} \leq -4$ , we have  $k_{f^{\hat{a}}S} \leq -4$ . Because  $g$  attains maximum at the point  $x \in D(0, r)$ ,  $\log(g)$

also attains maximum at the same point. Hence,  $\Delta \log(g(x)) \leq 0$ . But,

$$\begin{aligned} \Delta \log(g(x)) &= \Delta \log f^{\hat{a}} \mathcal{S}(x) - \Delta \log r_r(x) = \\ &= -k_{f^{\hat{a}} \mathcal{S}}(x) \cdot (f^{\hat{a}} \mathcal{S}(x))^2 + k_{r_r}(x) \cdot (r_r(x))^2 \geq 4 \left( (f^{\hat{a}} \mathcal{S}(x))^2 - (r_r(x))^2 \right) \end{aligned} \quad (5)$$

This shows that  $\frac{f^{\hat{a}} \mathcal{S}(x)}{r_r(x)} \leq 1 \Rightarrow M \leq 1$ . But,  $M \leq 1 \Rightarrow g \leq 1$  on  $D(0, r)$  and letting  $r \rightarrow 1^-$  completes the proof.

**Remark.**

The usual version of Schwarz Lemma follows immediately from the above. Let  $\Omega = \mathbb{D}$  and  $\mathcal{S} = r$ , the Poincaré metric. If  $f : \mathbb{D} \rightarrow \mathbb{D}$  be a holomorphic map fixing the origin then applying above result yields

$$f^{\hat{a}} r(0) \leq r(0) \Rightarrow |f'(0)| \cdot r(f(0)) \leq r(0) \Rightarrow |f'(0)| \leq 1.$$

Corollary 2.9.

Let the domain  $\Omega \subseteq \mathbb{C}$  be equipped with a conformal metric  $\mathcal{S}$  whose curvature is bounded above by a negative constant  $-B$ . Then every holomorphic function  $f : D(0, r) \rightarrow \Omega$  satisfies

$$f^{\hat{a}} \mathcal{S}(z) \leq \frac{2}{\sqrt{B}} \cdot r_r(z) \quad \forall z \in D(0, r)$$

where  $r_r$  is a conformal metric on  $D(0, r)$  defined in Example 2.5.

*Proof:*

We follow the same construction as the above theorem to prove this corollary. Let  $0 < a < r$  and define a conformal metric  $r_a(z) = \frac{a}{a^2 - |z|^2}$  on  $D(0, a)$ , then  $r_a(z)$  has a constant

curvature  $-4$ . Define  $g(z) = \frac{f^{\hat{a}} \mathcal{S}}{\frac{2}{\sqrt{B}} \cdot r_a(z)}$  on  $D(0, a)$ .

Following the same arguments, we come to the conclusion that  $\frac{f^{\hat{a}} \mathcal{S}(x)}{\frac{2}{\sqrt{B}} \cdot r_a} \leq 1$  for all  $x \in D(0, a)$

and letting  $a \rightarrow r^-$ , we finally get that  $f^{\hat{a}} \mathcal{S}(z) \leq \frac{2}{\sqrt{B}} \cdot r_r(z) \quad \forall z \in D(0, r)$ .

Before we see another application of curvature in proving Liouville's Theorem, we state and prove another Lemma.

Lemma 2.10.

Let  $\Omega \subseteq \mathbb{C}$  be a open set equipped with a metric  $\mathcal{S}$  having the property that its curvature,  $k_{\mathcal{S}}$  satisfies  $k_{\mathcal{S}}(z) \leq -B < 0 \quad \forall z \in \Omega$ , where  $B$  is some positive constant, then every entire  $f$  from  $\mathbb{C}$  to  $\Omega$  must be constant.

*Proof:*

We know that  $D(0, r)$  is equipped with the metric  $r_r$  defined by  $r_r(z) = \frac{r}{r^2 - |z|^2}$  for every  $r > 0$ . The corollary of Schwarz Lemma about curvature implies that for any fixed  $z \in D(0, r)$ ,

$f^{\hat{a}}\mathcal{S}(z) \leq \frac{2}{\sqrt{B}} \cdot r_r(z)$ . Letting  $r \rightarrow \infty$  gives that  $f^{\hat{a}}\mathcal{S}(z) \leq 0$ . But,  $f^{\hat{a}}\mathcal{S}(z)$  is a pullback of  $\mathcal{S}$

under  $f$ , and hence a conformal metric on  $\mathbf{C}$ , so  $f^{\hat{a}}\mathcal{S}(z) = 0 \Rightarrow \mathcal{S}(f(z)) \cdot |f'(z)| = 0$ .

But, this is possible only when  $f'(z) = 0$ . Since  $z$  is arbitrary,  $f'(z) = 0$  implies that  $f$  is constant.

**Theorem 2.11 (Liouville's Theorem).**

*Any bounded entire function is constant.*

*Proof:*

This theorem is an immediate consequence of the above Lemma. Let  $f$  be entire and bounded.

Since  $f$  is bounded, there exists  $M > 0$  such that  $|f(z)| \leq M \quad \forall z \in \mathbf{C}$ . Define  $g(z) = \frac{1}{M} \cdot f(z)$ ,

then range of  $g$  is contained in  $\mathbf{D}$ . But, Poincaré metric on  $\mathbf{D}$  has constant curvature  $-4$ . Thus, all the conditions of the above Lemma are satisfied and hence  $f$  is constant.

We have seen the easy proofs of two important theorems in complex analysis by using the theory we developed about curvature. Next, we will prove Picard's Little Theorem as another application. This theorem says that the hypothesis "bounded" in Liouville's theorem may be weakened considerably, yet the same conclusion may be drawn. So, without being bounded, an entire function need only omit a segment from its values for it to be a constant. If  $f(z) = e^z$  then  $f$  is entire and nonconstant and it assumes all complex values except zero. The very important result discovered by Picard says "a nonconstant entire function can not omit two values". Before we give the proof of this striking result, we state and prove a Lemma.

**Lemma 2.12.**

*Let  $\Omega$  be a domain such that  $\mathbf{C} \setminus \Omega$  contains at least two points. Then  $\Omega$  admits a metric  $m$  for which there exists a positive constant  $B$  such that the curvature  $k_m$  of  $m$  satisfies  $k_m \leq -B < 0$ .*

*Proof:*

Assume that the two omitted points are 0 and 1 and define a metric on  $\mathbf{C}_{0,1} \equiv \mathbf{C} \setminus \{0,1\}$  by

$$m(z) = \left[ \frac{(1+|z|^{1/3})^{1/2}}{|z|^{5/6}} \right] \cdot \left[ \frac{(1+|z-1|^{1/3})^{1/2}}{|z-1|^{5/6}} \right].$$

Then  $m$  is positive and smooth on  $\mathbf{C}_{0,1}$ . We now calculate the curvature of  $m$  at  $z \in \mathbf{C}_{0,1}$ .

$$k_m(z) = \frac{-\Delta \log(m(z))}{(m(z))^2} = \frac{-\Delta \log \left[ \frac{(1+|z|^{1/3})^{1/2}}{|z|^{5/6}} \right] - \Delta \log \left[ \frac{(1+|z-1|^{1/3})^{1/2}}{|z-1|^{5/6}} \right]}{(m(z))^2} \quad (6)$$

But,  $\Delta(\log |z|^{5/6}) = \frac{5}{12} \Delta(\log |z|^2) = 0$  away from the origin. Similarly, away from  $z=1$ ,

$$\Delta(\log |z-1|^{5/6}) = \frac{5}{12} \Delta(\log |z-1|^2) = 0.$$

Now,

$$-\Delta \log \left[ \frac{(1+|z|^{1/3})^{1/2}}{|z|^{5/6}} \right] = -\frac{1}{2} \Delta \log(1+|z|^{1/3}) = -2 \frac{\partial^2}{\partial z \partial \bar{z}} \log(1+[z \cdot \bar{z}]^{1/6}) = -\frac{1}{18} \frac{1}{|z|^{5/3} (1+|z|^{1/3})^2} \quad (7)$$

Similarly, we have

$$-\Delta \log \left[ \frac{(1+|z-1|^{1/3})^{1/2}}{|z-1|^{5/6}} \right] = -\frac{1}{18} \frac{1}{|z-1|^{5/3} (1+|z-1|^{1/3})^2}. \quad (8)$$

so, from equations (7) and (8), we get

$$k_m(z) = -\frac{1}{18} \left[ \frac{|z-1|^{5/3}}{(1+|z|^{1/3})^3 (1+|z-1|^{1/3})} + \frac{|z|^{5/3}}{(1+|z-1|^{1/3})^3 (1+|z|^{1/3})} \right] \quad (9)$$

If we observe equation (9), we find the following facts.

- 1).  $k_m(z) < 0$  for all  $z \in \mathbb{C}_{0,1}$ ;
- 2).  $\lim_{z \rightarrow 0} k_m(z) = -1/36$ ;
- 3).  $\lim_{z \rightarrow 1} k_m(z) = -1/36$ ;
- 4).  $\lim_{z \rightarrow \infty} k_m(z) = -\infty$ .

These facts clearly imply that there exists a positive constant  $B$  such that  $k_m(z) \leq -B$ .

Now we state and prove Picard's little theorem. The proof is a direct consequence of two results we proved earlier.

Theorem 2.13 (Picard's little theorem).

Let  $f: \mathbb{C} \rightarrow \Omega$  be an entire function where  $\Omega \subseteq \mathbb{C}$  is a planar domain. If  $\mathbb{C} \setminus \Omega$  contains at least two points, then  $f$  is constant.

*Proof:*

Since  $\mathbb{C} \setminus \Omega$  contains at least two points, by Lemma 2.12  $\Omega$  admits a metric  $m$  such that there exists a positive constant  $B$  such that  $k_m(z) \leq -B$  for all  $z$ . But the lemma 2.10 just before the Liouville's theorem implies that any such entire function  $f$  is constant.

We close this section with two more definitions and the statement of the general version of Montel Theorem.

Definition 2.14 (normally convergent and compactly divergent sequence).

A sequence of functions in  $\Omega$  is said to be normally convergent if it converges uniformly on compact subsets of  $\Omega$ . We say that the sequence is compactly divergent if it converges to  $\infty$ , uniformly on compact sets.

Definition 2.15 (normal family).

A family  $F$  of  $\hat{\mathbb{C}}$ -valued holomorphic functions on a domain  $\Omega \subseteq \mathbb{C}$ ,  $\hat{\mathbb{C}}$  represents the Riemann sphere, is called a normal family if every sequence in  $F$  has a subsequence which is either normally convergent or compactly divergent.

Theorem 2.16 (Montel).

Let  $\Omega = \hat{\mathbb{C}} \setminus \{P, Q, R\}$ , where  $P, Q$  and  $R$  are distinct points in  $\hat{\mathbb{C}}$ . If  $F$  is the family of the holomorphic functions taking values in  $\Omega$ , then  $F$  is a normal family.

### 3. Carathéodory and Kobayashi metrics

In this section, we introduce two more metrics, the Carathéodory metric and the Kobayashi metric. These metrics are invariant under conformal mappings and motivated by the extremal problem that arises from the Riemann mapping theorem. First we define these two metrics, give examples, and explain some of their properties. Later we explore the interaction between them.

### 3.1. The Carathéodory metric.

Let  $\Omega$  be any domain in  $\mathbb{C}$  and  $\mathbb{D}$  the unit disc. If  $P \in \Omega$ , we define

$$F_{\Omega\mathbb{D}}(P) = \{\text{all holomorphic functions } f: \Omega \rightarrow \mathbb{D} \text{ s.t. } f(P) = 0\}.$$

The Carathéodory metric for  $\Omega$  at  $P$  is denoted by  $F_C^\Omega(P)$  and is defined by

$$F_C^\Omega(P) = \sup\{|f'(P)| : f \in F_{\Omega\mathbb{D}}(P)\}.$$

Clearly,  $F_C^\Omega(P) < \infty$ . This is because of the Cauchy estimates. At  $P \in \Omega$ ,  $|f'(P)| < (1/r)$  on  $D(0, r)$  for every  $r > 0$  such that  $D(0, r) \subset \Omega$ . Also,  $F_C^\Omega(P) \geq 0$ . We now consider the two following cases.

#### Case 1.

If  $\Omega$  is bounded then  $F_C^\Omega(P) > 0 \quad \forall P \in \Omega$ . Since  $\Omega$  is bounded, there exists  $r > 0$  s.t.  $\Omega \subset D(0, r)$ .

For any  $x \in D(0, r)$ , define  $f(x) = \frac{x-P}{2r}$ . Then  $f$  is holomorphic,  $f(P) = 0$  and

$$f'(x) = \frac{1}{2r} \Rightarrow |f'(P)| = \frac{1}{2r}. \text{ Therefore, by definition, } F_C^\Omega(P) \geq |f'(P)| = \frac{1}{2r} > 0.$$

**Case 2.** If  $\Omega$  is unbounded then  $F_C^\Omega$  may degenerate. If we take  $\Omega = \mathbb{C}$ , then for  $P \in \mathbb{C}$ ,  $f \in F_{\mathbb{C}\mathbb{D}}(P)$  implies that  $f$  is entire and bounded. By Liouville's Theorem,  $f$  is constant, so  $f' = 0$ . Hence  $F_C^\Omega = 0$ .

In Proposition 1.15 given in [6], we proved that every holomorphic function  $f: \mathbb{D} \rightarrow \mathbb{D}$  is distance decreasing in the Poincaré metric. Now, we will show that the Carathéodory metric generalizes the above theorem to arbitrary domains  $\Omega$ .

#### Proposition 3.1 (distance decreasing property of the Carathéodory metric).

Let  $\Omega_1$  and  $\Omega_2$  be domains in  $\mathbb{C}$ . Let  $r_1$  and  $r_2$  be the Carathéodory metric on  $\Omega_1$  and  $\Omega_2$  respectively. If  $h: \Omega_1 \rightarrow \Omega_2$  is holomorphic then  $h^\hat{a} r_2(z) \leq r_1(z)$ ,  $\forall z \in \Omega_1$ .

*Proof:* We fix  $P \in \Omega_1$ . So,  $h(P) \in \Omega_2$ . If  $f \in F_{\Omega_2\mathbb{D}}(h(P))$  then  $f(h(P)) = 0$ . But,

$$(f \circ h)(P) = f(h(P)) = 0 \Rightarrow f \circ h \in F_{\Omega_1\mathbb{D}}(P).$$

Now, the Carathéodory metric for  $\Omega_1$  at  $P$  is  $F_C^{\Omega_1}(P) \geq |(f \circ h)'(P)| = |f'(h(P))| \cdot |h'(P)|$ .

Taking supremum over  $f \in F_{\Omega_2\mathbb{D}}(h(P))$ , we get that  $F_C^{\Omega_1}(P) \geq F_C^{\Omega_2}(h(P)) \cdot |h'(P)|$ . But by hypothesis,  $r_2 = F_C^{\Omega_2}$  and  $r_1 = F_C^{\Omega_1}$  and the above equation implies that

$$r_1(P) \geq r_2(h(P)) \cdot |h'(P)| = h^\hat{a} r_2(P).$$

Since  $P$  is arbitrary, we conclude that  $h^\hat{a} r_2(z) \leq r_1(z)$ ,  $\forall z \in \Omega_1$ .

#### Corollary 3.2.

If  $g: [0, 1] \rightarrow \Omega_1$  is a piecewise continuously differentiable curve then  $l_{r_2}(h \circ g) \leq l_{r_1}(g)$ .

The proof of this Corollary is similar to the proof of Prop 1.15(b) given in [6]. We just need to replace  $\mathbb{D}$  by  $\Omega_1$ .

#### Corollary 3.3.

If  $P, Q \in \Omega_1$ , then  $d_{r_2}(h(P), h(Q)) \leq d_{r_1}(P, Q)$ . The proof of this is an immediate consequence of the above Corollary.

**Corollary 3.4.**

If  $h$  is a conformal map from  $\Omega_1$  to  $\Omega_2$  then  $h$  is an isometry of  $(\Omega_1, r_1)$  to  $(\Omega_2, r_2)$ .

*Proof:* To prove this corollary, we have to show that  $h^{\hat{a}} r_2(z) = r_1(z)$  for every  $z \in \Omega_1$ . From the proposition 3.1, we have that

$$h^{\hat{a}} r_2(z) \leq r_1(z). \quad (10)$$

For the reverse inequality, note that  $h^{-1}$  is also conformal because  $h$  is conformal by hypothesis. So  $h^{-1}$  is distance decreasing from  $(\Omega_2, r_2)$  to  $(\Omega_1, r_1)$  i.e.

$$(h^{\hat{a}})^{-1} r_1(h(z)) \leq r_2(h(z)).$$

But,

$$(h^{\hat{a}})^{-1} r_1(h(z)) = r_1(h^{-1}(h(z))) \cdot |(h^{-1})'(h(z))| = r_1(z) \cdot \left| \frac{1}{h'(z)} \right| \leq r_2(h(z))$$

This implies that

$$r_1(z) \leq r_2(h(z)) \cdot |h'(z)| \Rightarrow r_1(z) \leq h^{\hat{a}} r_2(z) \quad (11)$$

Combining equation (10) and (11) gives the desired result.

We may now ask a question: what is the Carathéodory metric on the unit disc? The answer to this question is given here in the form of the proposition.

**Proposition 3.5.**

The Carathéodory metric on the unit disc and the Poincaré metric are equal to each other.

*Proof:*

First, we calculate the Carathéodory metric at the origin of the unit disc,  $\mathbb{D}$ . Let  $f \in F_{\mathbb{D}\mathbb{D}}(0)$ . Then by Schwarz lemma,  $|f'(0)| \leq 1$ . But, if  $f(z) = z$  then  $f \in F_{\mathbb{D}\mathbb{D}}(0)$  and  $f'(0) = 1$ . So, by the definition of Carathéodory metric, we get that  $F_C^{\mathbb{D}}(0) = 1$ .

By corollary 3.4, every conformal map of the disc is an isometry of the Carathéodory metric. Hence by theorem 1.21 from [6],  $F_C^{\mathbb{D}}$  is a constant multiple of Poincaré metric  $r$ . In fact,  $F_C^{\mathbb{D}}(z) = F_C^{\mathbb{D}}(0) \cdot r(z)$  but,  $F_C^{\mathbb{D}}(0) = 1$  implies that  $F_C^{\mathbb{D}}(z) = r(z)$  for every  $z \in \mathbb{D}$ .

Next, we introduce the Kobayashi metric and explain some of its properties.

**3.2. The Kobayashi metric.**

Let  $\Omega$  be any domain in  $\mathbb{C}$  and  $\mathbb{D}$  the unit disc. If  $P \in \Omega$ , we define

$$F_{\mathbb{D}\Omega}(P) = \{ \text{all holomorphic functions } f: \mathbb{D} \rightarrow \Omega \text{ s.t. } f(0) = P \}.$$

The Kobayashi metric for  $\Omega$  at  $P$  is denoted by  $F_K^{\Omega}(P)$  and is defined by

$$F_K^{\Omega}(P) = \inf \left\{ \frac{1}{|f'(0)|} : f \in F_{\mathbb{D}\Omega}(P) \right\}.$$

Clearly,  $F_K^{\Omega}(P) < \infty$ . If we take  $f(z) = rz + P$ , then  $f$  is holomorphic on  $\mathbb{D}$ ,  $f(\mathbb{D}) \subset \Omega$  if  $r$  is small enough,  $f(0) = P$  and  $f'(0) = r$ . Hence  $F_K^{\Omega}(P) \leq \frac{1}{r}$  for  $r > 0$  is small enough. Also,

$F_K^{\Omega}(P) \geq 0$ . To prove this, we first prove the following proposition.

**Proposition 3.6.**

$F_C^{\Omega}(P) \leq F_K^{\Omega}(P)$  for all  $P$  in  $\Omega$ .

*Proof:*

Let  $f \in F_{\Omega D}(P)$  and  $f \in F_{D\Omega}(P)$  with  $f'(0) \neq 0$ . Then  $f \circ f: D \rightarrow D$  and  $(f \circ f)(0) = f(f(0)) = f(P) = 0$ .

So, by Schwarz lemma, we have  $|(f \circ f)'(0)| \leq 1 \Rightarrow |f'(P)| \leq \frac{1}{|f'(0)|}$ . If we take supremum over all  $f \in F_{\Omega D}(P)$ , we get that  $F_C^\Omega(P) \leq \frac{1}{|f'(0)|}$ . Now taking the infimum over all  $f \in F_{D\Omega}(P)$  gives  $F_C^\Omega(P) \leq F_K^\Omega(P)$ . This completes the proof of the proposition.

**Case 1.**

If  $\Omega$  is bounded, we saw right after the definition of Carathéodory metric that  $F_C^\Omega(P) > 0$ . The above proposition now clearly implies that  $F_K^\Omega(P) > 0$  for all  $P$  in  $\Omega$ .

**Case 2.**

If  $\Omega = \mathbb{C}$  then  $F_K^\Omega \equiv 0$ . For any given  $P \in \mathbb{C}$ , define  $f$  by  $f(z) = P + rz$  for any  $r > 0$  then  $f \in F_{D\Omega}(P)$  and  $F_K^\Omega(P) \leq \frac{1}{|f'(0)|} = \frac{1}{r}$ .

Letting  $r \rightarrow \infty$ , we get that  $F_K^\Omega(P) = 0$ . Since  $P \in \mathbb{C}$  is arbitrary,  $F_K^\Omega \equiv 0$ .

**Case 3.**

If  $\Omega = \mathbb{C} \setminus \{0\}$  then  $F_K^\Omega \equiv 0$ . Fix  $P \in \Omega$ . If  $e^Q = P$ , define  $f_n(z) = e^{nz+Q}$  then  $f_n \in F_{D\Omega}(P)$  and  $F_K^\Omega(P) \leq \frac{1}{|f_n'(0)|} = \frac{1}{n|P|}$ .

Letting  $n \rightarrow \infty$ , we get that  $F_K^\Omega(P) = 0$  since  $0 \notin \Omega$ . As  $P \in \Omega$  is arbitrary,  $F_K^\Omega \equiv 0$ . More generally, if  $\Omega = \mathbb{C} \setminus \{P\}$  for some  $P \in \mathbb{C}$  then  $F_K^\Omega \equiv 0$ .

The following proposition explains the distance decreasing property of a Kobayashi metric.

Proposition 3.7 (distance decreasing property of Kobayashi metric).

Let  $\Omega_1$  and  $\Omega_2$  be domains in  $\mathbb{C}$  which are equipped with the Kobayashi metric  $r_1$  and  $r_2$  respectively. If  $h: \Omega_1 \rightarrow \Omega_2$  is holomorphic then  $h^{\hat{a}} r_2(z) \leq r_1(z)$ ,  $\forall z \in \Omega_1$ .

*Proof:*

We fix  $P \in \Omega_1$ . So,  $h(P) \in \Omega_2$ . If  $f \in F_{D\Omega_1}(P)$  then  $f(0) = P$ . But,

$$(h \circ f)(0) = h(f(0)) = h(P) \Rightarrow f \circ h \in F_{D\Omega_2}(h(P)).$$

Now, the Kobayashi metric for  $\Omega_2$  at  $h(P)$  is  $F_K^{\Omega_2}(h(P)) \leq \frac{1}{|(h \circ f)'(0)|} = \frac{1}{|h'(f(0))| \cdot |f'(0)|}$ .

Taking the infimum over  $f \in F_{D\Omega_1}$ , we get that  $F_K^{\Omega_2}(h(P)) \leq \frac{F_K^{\Omega_1}(P)}{|h'(P)|}$ . But by hypothesis,

$$r_2 = F_C^{\Omega_2}, r_1 = F_C^{\Omega_1} \text{ and the above equation implies that } r_2(h(P)) \cdot |h'(P)| \leq r_1(P) \Rightarrow h^{\hat{a}} r_2(P) \leq r_1(P).$$

Since  $P$  is arbitrary, we conclude that  $h^{\hat{a}} r_2(z) \leq r_1(z) \quad \forall z \in \Omega_1$ .

Corollary 3.8.

If  $g: [0,1] \rightarrow \Omega_1$  is a piecewise continuously differentiable curve then  $l_{r_2}(h \circ g) \leq l_{r_1}(g)$ .

The proof of this Corollary is similar as the proof of Corollary 3.2.

Corollary 3.9.

If  $P_1, P_2 \in \Omega_1$  then  $d_{r_2}(h(P_1), h(P_2)) \leq d_{r_1}(P_1, P_2)$ .

The proof of this is the immediate consequence of the above Corollary.

Corollary 3.10.

If  $h$  is a conformal map from  $\Omega_1$  to  $\Omega_2$  then  $h$  is an isometry of  $(\Omega_1, r_1)$  to  $(\Omega_2, r_2)$ .

The proof of this corollary is same as the proof of corollary 3.4.

Now we turn our attention to the question: What is the Kobayashi metric on unit disc?

Proposition 3.11.

The Kobayashi metric on the unit disc,  $\mathbf{D}$  coincides with the Poincaré metric.

*Proof:*

Let  $r$  be the Poincaré metric on  $\mathbf{D}$ . We have to show that  $F_K^{\mathbf{D}}(z) = r(z)$  for every  $z \in \mathbf{D}$ .

( $\geq$ ) For any  $P \in \mathbf{D}$ , we have by Prop 3.5 and Prop 3.6 that  $F_K^{\mathbf{D}}(P) \geq F_C^{\mathbf{D}}(P) = r(P)$ .

In particular,

$$F_K^{\mathbf{D}}(0) \geq r(0). \quad (12)$$

( $\leq$ ) Let  $f(z) = z$ , then  $f \in F_{\mathbf{D}\mathbf{D}}(0)$  and  $f'(0) = 1$ . Hence, by definition,

$$F_K^{\mathbf{D}}(0) \leq \frac{1}{|f'(0)|} = 1 = r(0). \quad (13)$$

Hence, equation (12) and (13) imply that  $F_K^{\mathbf{D}}(0) = r(0) = 1$ .

By Corollary 3.10, every conformal map of the unit disc is an isometry of the Kobayashi metric and hence by theorem 1.21 described in [6], the Kobayashi metric is a constant multiple of the Poincaré metric. In fact,  $F_K^{\mathbf{D}}(P) = F_K^{\mathbf{D}}(0) \cdot r(P) \Rightarrow F_K^{\mathbf{D}}(P) = r(P)$  for every  $P \in \mathbf{D}$ .

This result proves the Proposition.

Now we focus on the metric version of Riemann mapping Theorem.

Theorem 3.12.

Let  $\Omega$  be a domain in  $\mathbf{C}$  then  $\Omega$  is conformally equivalent to the unit disc if and only if  $F_C^{\Omega}(P) = F_K^{\Omega}(P) \neq 0$  for some  $P \in \Omega$ .

*Proof:*

( $\Rightarrow$ ) Suppose that the domain  $\Omega$  is conformally equivalent to the unit disc,  $\mathbf{D}$ . Then there exist a conformal map, say  $f$  from  $\Omega$  to  $\mathbf{D}$ . But, by corollary 3.4,  $f$  is an isometry in Carathéodary metric. So, for any point  $P \in \Omega$ , we have

$$F_C^{\Omega}(P) = f^{\hat{a}} F_C^{\mathbf{D}}(P) \quad (14)$$

By proposition 3.5,  $F_C^{\mathbf{D}}(P) = r(P)$  where  $r$  is the Poincaré metric on  $\mathbf{D}$ . Again, by the proposition 3.11,  $r(P) = F_K^{\mathbf{D}}(P)$ . Thus, the equation (14) turns to  $F_C^{\Omega}(P) = f^{\hat{a}} F_K^{\mathbf{D}}(P)$ .

Since  $f$  is conformal,  $f$  is an isometry in Kobayashi metric as well and hence  $f^{\hat{a}} F_K^{\mathbf{D}}(P) = F_K^{\Omega}(P)$ .

Finally we get,  $F_C^{\Omega}(P) = F_K^{\Omega}(P)$  which gives the proof for one direction.

( $\Leftarrow$ ) Suppose that  $F_C^{\Omega}(P) = F_K^{\Omega}(P)$  for some point  $P \in \Omega$ . We have to show that  $\Omega$  is conformally equivalent to  $\mathbf{D}$ .

Now, choose  $\{f_j\} \subseteq F_{\Omega\mathbf{D}}(P)$  such that  $|f_j'(P)| \rightarrow F_C^{\Omega}(P)$  and choose  $\{f_j\} \subseteq F_{\mathbf{D}\Omega}(P)$  such that

$$\frac{1}{|f'_j(0)|} \rightarrow F_K^\Omega(P).$$

Since  $|f_j| \leq 1$  for every  $j$ , we may extract a subsequence  $f_{j_k}$  converging to a normal limit  $g$ . By Montel theorem,  $\{f_j\}$  forms a normal family because  $\mathbb{C} \setminus \Omega$  contains at least two points (otherwise  $F_K^\Omega \equiv 0$  contradicting the given hypothesis). Let's extract a subsequence  $f_{j_k}$  converging normally to the limit  $y$ .

Let  $h_{j_k} = f_{j_k} \circ f_{j_k}$ .

Passing to another subsequence, say  $h_{j_{k_l}}$ , we may suppose that  $h_{j_{k_l}}$  converges normally to a limit  $t$ . After renumbering, we call this last sequence  $h_l = f_l \circ f_l$ .

Thus, we have  $h_l \rightarrow t$ . Since  $t(0)=0$ ,  $t$  maps  $\mathbb{D}$  to  $\mathbb{D}$ . By Cauchy estimates, if the sequence of holomorphic functions converges normally, so does the sequence of its derivatives. So,

$$t'(0) = \lim_{l \rightarrow \infty} |h'_l(0)| = \lim_{l \rightarrow \infty} |(f_l \circ f_l)'(0)| = \lim_{l \rightarrow \infty} |f'_l(P)| \cdot |f'_l(0)| = F_C^\Omega(P) \cdot \frac{1}{F_K^\Omega(P)} = 1.$$

Since  $t : \mathbb{D} \rightarrow \mathbb{D}$ ,  $t$  is holomorphic,  $t(0)=0$  and  $t'(0)=1$ , by Schwarz lemma  $t$  is rotation. So, there exist  $m \in \mathbb{R}$  such that  $t(z) = mz$ . But  $m \cdot z = t(z) = \lim_{l \rightarrow \infty} (f_l \circ f_l(z))$ .

With some modification by a constant, assume that  $m = 1$ . Then we get

$$z = t(z) = \lim_{l \rightarrow \infty} (f_l \circ f_l(z)) = g \circ y(z).$$

Since the limit  $t$  is surjective, so is  $g$ . As  $y'(0) \neq 0$ , the function  $y$  is a non-constant holomorphic function, hence its image is open. The image of  $y$  is also closed in the relative sense. But, the image of  $y$  is nonempty and  $\Omega$  is connected, it follows that the image of  $y$  equals  $\Omega$ , hence  $y$  is surjective. Since  $t$  is also injective,  $g$  must be injective. So, we conclude that  $g$  is the desired conformal map of  $\Omega$  to  $\mathbb{D}$  and hence  $\Omega$  is conformally equivalent to the unit disc,  $\mathbb{D}$ .

#### 4. The completeness of the Carathéodory and the Kobayashi metrics

In this section, we discuss the completeness of the Carathéodory and the Kobayashi metrics on a reasonable class of domains. To reach our goal, we begin by defining peak point and a peak function for a domain.

**Definition 4.1 (Peak point and peak function for a domain).**

Let  $\Omega \subseteq \mathbb{C}$  be a domain and  $P \in \partial\Omega$ . We say  $P$  is a peak point for  $\Omega$  if there exists a function  $f \in H(\Omega) \cap C(\overline{\Omega})$  such that  $f(P)=1$ ,  $|f(z)| < 1$  for  $z \in \overline{\Omega} \setminus \{P\}$ . We call  $f$  a peak function for  $\Omega$  at  $P$ . Now we will see one example of a peak function.

**Example 4.2.**

Let  $P \in \partial\mathbb{D}$  and define the function  $f$  on  $\mathbb{D}$  by  $f(z) = \exp(\overline{P}z - 1)$  then  $f$  is a peak function for  $\mathbb{D}$ .

The reason is clear. Since  $P \in \partial\mathbb{D}$ , we have  $|P|=1$ , and  $|\overline{P}|=1$ . So,

$$f(P) = \exp(\overline{P}P - 1) = \exp(|P|^2 - 1) = 1. \text{ For } |z| \leq 1, z \neq P,$$

$$|f(z)| = |\exp(\overline{P}z - 1)| = \exp(\operatorname{Re}(\overline{P}z - 1)) \tag{15}$$

$$\text{but } \operatorname{Re}(\overline{P}z - 1) = \operatorname{Re}(\overline{P}z) - 1 < 0$$

Then by equation 15, we have that  $|f(z)| < 1$ . Also,  $f$  is holomorphic on  $\mathbb{D}$  and continuous up to

the boundary  $\partial D$ . Hence  $f$  is a peak function for  $D$  and  $P$  is the peak point.

**Theorem 4.3.**

Suppose  $\Omega \subseteq \mathbb{C}$  is a bounded domain such that each  $P \in \partial\Omega$  is a peak point for  $\Omega$ . Then  $\Omega$  is complete in the Carathéodory and Kobayashi metrics.

*Proof:*

Let  $m = F_C^\Omega$ .

**Claim:**

If  $\{p_j\} \subset \Omega$  and  $p_j \rightarrow p \in \partial\Omega$ , then  $d_m(p_j, p_0) \rightarrow \infty$  for every  $p_0 \in \Omega$ .

*proof of the claim:*

Let  $f$  be a peak function for  $\Omega$  at  $P$ . Then by proposition 3.1, the distance decreasing property of the Carathéodory metric, we have  $f^{\hat{a}} r \leq m$ , which implies, by corollary 3.3, that  $d_r(f(p_j), f(p_0)) \leq d_m(p_j, p_0)$ . But,  $d_m(p_j, p_0) \geq d_r(f(p_j), f(p_0)) \rightarrow \infty$  as  $j \rightarrow \infty$  by theorem 1.21 given in NK.

This proves the claim.

Assume that  $\{p_j\}$  is cauchy in  $d_m$ . Then  $\{p_j\}$  is bounded in  $d_m$ . But, by the claim, there exists  $e$  such that

$$dist(p_j, \partial\Omega) \geq e > 0. \tag{16}$$

By equation (16), because the sequence is cauchy in  $d_m$ , the sequence is cauchy in the Euclidean metric, and the sequence  $\{p_j\}$  converges to some point  $q \in \Omega$ . But, the equation (16) implies that the point  $q \in \Omega$ . Now, proposition 1.9 given in [6] shows that the domain  $\Omega$  is complete in the Carathéodory metric. Since the Kobayashi metric is always greater than the Carathéodory metric, any sequence that is Cauchy in Kobayashi metric is also Cauchy in Carathéodory metric, so the theorem is also true for Kobayashi metric.

Let's ask a question. When is a point  $P \in \partial\Omega$  a peak point for the domain  $\Omega$ ? We prove one proposition that gives the answer of the above question.

**Proposition 4.4.**

Let  $\Omega$  be a bounded domain and  $P \in \partial\Omega$  such that there exists a disc  $D_P = D(Q, s)$  with  $\overline{\Omega} \cap \overline{D_P} = \{P\}$ . Then  $P$  is a peak point for  $\Omega$ .

*Proof:*

Since  $P \in \partial\Omega$  and  $\Omega$  is open, we have that  $P \in \partial D_P$ . Now, we define  $f(z) = \frac{P-Q}{z-Q}$ . Then  $f$  is

holomorphic on  $\Omega$  and continuous up to the boundary of  $\Omega$ . Also,  $f(P) = \frac{P-Q}{P-Q} = 1$ , and for

$z \in \overline{\Omega} \setminus \{P\}$ , we have that  $|z-Q| > s \Rightarrow |f(z)| < 1$ . This clearly shows that  $f$  is a peak function and  $P$  is a peak point for  $\Omega$ .

Next, we explain one example to show why certain point is not a peak point for a certain domain.

**Example 4.5.**

Let  $\Omega = \mathbb{D} \setminus \{0\}$ . Then  $0$  is not a peak point for  $\Omega$ .

**Reason:**

If  $0$  is a peak point for  $\Omega$ , then there exists  $f \in H(\Omega) \cap C(\overline{\Omega})$  such that  $f(0)=1$  and  $f(z)<1 \forall z \in \Omega$ . By Riemann removable singularity theorem, there exists an analytic function  $\hat{f} : \mathbb{D} \rightarrow \mathbb{D}$  such that  $\hat{f}(z) = f(z)$  for  $z \neq 0$  and  $\hat{f}(0) = \lim_{z \rightarrow 0} f(z) = 1$ . But the Maximum modulus

theorem implies that the function  $f$  is constant, showing  $f$  is constant, which is a contradiction. Thus, 0 can not be a peak point for  $\Omega$ .

**Definition 4.6.**

Let  $\Omega \subseteq \mathbb{C}$  be a bounded domain. We say  $\Omega$  has a  $C^2$  boundary if there exists neighbourhood  $\Lambda$  of  $\partial\Omega$  and a function  $r \in C^2_{\mathbb{R}}(\Lambda)$  such that  $\Lambda \cap \Omega = \{z \in \Lambda : r(z) < 0\}$  and  $\nabla r(z) \neq 0$  on  $\Lambda$ .

Then Implicit Function Theorem gives  $\partial\Omega \cap \Lambda = \{z \in \Lambda : r(z) = 0\}$  and  $\Lambda \setminus \overline{\Omega} = \{z \in \Lambda : r(z) > 0\}$ .

Let's state, without proof, an important geometric fact about domains having  $C^2$  boundary.

**Geometric Fact:**

Suppose  $\Omega \subset \subset \mathbb{C}$  has  $C^2$  boundary. Then for every  $P \in \partial\Omega$ , there exists a disc  $D_P$  as in the proposition 4.4.

Now, we turn to the main result of this section.

**Theorem 4.7.**

If  $\Omega \subseteq \mathbb{C}$  is a bounded domain with a  $C^2$  boundary then  $\Omega$  is complete in Carathéodory and Kobayashi metrics.

*Proof:*

By the above geometric fact, since  $\Omega$  is a bounded domain with a  $C^2$  boundary, there exists a disc  $D_P$  with  $\overline{\Omega} \cap \overline{D_P} = \{P\}$  for every  $P \in \partial\Omega$ . Then by proposition 4.4,  $P$  is a peak point for  $\Omega$ .

This means that each point  $P \in \partial\Omega$  is a peak point for  $\Omega$ . Hence, by theorem 4.3, we conclude that  $\Omega$  is complete in the Carathéodory and Kobayashi metrics.

### 5. Hyperbolicity

In this section, we discuss some concepts related to curvature and nondegeneracy of the Kobayashi metric. First, we start with the fact that the Carathéodory metric is never greater than the Kobayashi metric. As usual,  $r$  denotes the Poincaré metric on the unit disc  $\mathbb{D}$ .

**Theorem 5.1.**

Let  $\Omega$  be a domain and is equipped with a metric  $s$ . If every holomorphic function  $f : \mathbb{D} \rightarrow \Omega$  is distance decreasing from  $(\mathbb{D}, r)$  to  $(\Omega, s)$  then  $s \leq F_K^\Omega$ .

*Proof:*

Let  $P$  be in  $\Omega$ . If  $f \in F_{\mathbb{D}\Omega}(P)$  then by definition  $f : \mathbb{D} \rightarrow \Omega$  and  $f(0) = P$ . Since, by hypothesis,  $f$  is distance decreasing, we have  $f^*s(0) \leq r(0)$ . But  $r(0) = 1$ , so

$$f^*s(0) = s(f(0)) \cdot |f'(0)| = s(P) \cdot |f'(0)| \leq r(0) = 1$$

which implies that  $\frac{1}{|f'(0)|} \geq s(P)$ .

Taking the infimum over all  $f \in F_{\mathbb{D}\Omega}(P)$ , we get that  $s(P) \leq F_K^\Omega(P)$ . But,  $P \in \Omega$  is arbitrary so  $s(z) \leq F_K^\Omega(z) \quad \forall z \in \Omega$ .

We now define the hyperbolicity of a domain.

**Definition 5.2.**

A domain  $\Omega \subseteq \mathbb{C}$  is called hyperbolic if  $F_K^\Omega(P) > 0$  for any  $P \in \Omega$ .

Next, we prove the fact that a domain equipped with a metric  $s$  that has curvature bounded above by a negative constant is hyperbolic.

**Theorem 5.3.**

Let  $\Omega \subseteq \mathbb{C}$  be a domain equipped with a metric  $s$ . If there exists a positive constant  $B$  such that

the curvature  $k$  of the metric  $\mathcal{S}$  satisfies  $k \leq -B < 0$  then  $\Omega$  is hyperbolic.

*Proof:*

Let  $f : \mathbb{D} \rightarrow \Omega$  is holomorphic then by Corollary 2.9, we have,  $f^* \mathcal{S} \leq \frac{2}{\sqrt{B}} r$ , where  $r$  is the Poincaré metric on  $\mathbb{D}$ . If we suppose  $\mathcal{S} = \mathcal{S} \sqrt{B}/2$ , then the above equation reduces to the form  $f^* \mathcal{S} \leq r$ .

Applying theorem 5.1, we find that  $\mathcal{S} \leq F_K^\Omega \Rightarrow \frac{\sqrt{B}}{2} \cdot \mathcal{S} \leq F_K^\Omega$ .

Since  $\mathcal{S}$  is nondegenerate and the Kobayashi distance is bounded from below by a constant times  $\mathcal{S}$ , we conclude that  $F_K^\Omega$  is also nondegenerated and hence  $\Omega$  is hyperbolic by definition.

**Note:**

We saw just below the proof of proposition 3.6 that the Kobayashi metric degenerates for  $\mathbb{C}$  and punctured plane  $\mathbb{C} \setminus \{0\}$ . Thus, neither of them are hyperbolic. Theorem 3.22 shows that they can not possess metrics of strictly negative curvature. But, we proved in lemma 2.12 that if  $\mathbb{C} \setminus \Omega$  contains at least two points then  $\Omega$  admits a metric with strictly negative curvature. Hence, by theorem 3.22,  $\Omega = \mathbb{C} \setminus \{P_1, P_2, \dots, P_k\}, k \geq 2$  is hyperbolic if  $P_j \neq P_l$  when  $j \neq l$ .

Next, we define new terminology, describe it with some examples and connect hyperbolicity of a domain with the concept of normal families.

Definition 5.4.

Let  $\Omega \subseteq \mathbb{C}$  be a domain. Define  $F = \{f : \mathbb{D} \rightarrow \Omega \text{ such that } f \text{ is holomorphic}\}$ . Then  $\Omega$  is called *taut* if  $F$  is normal in the sense of definition 2.15.

Let's see one example of *taut* and one non-example.

Example 5.5.

*The unit disc,  $\mathbb{D}$  is taut.*

Reason: Let  $F = \{f : \mathbb{D} \rightarrow \mathbb{D} \text{ such that } f \text{ is holomorphic}\}$ . If  $\{f_j\} \subseteq F$  then  $|f_j| \leq 1$  for each  $j$ . By Montel's theorem, there exists a subsequence  $\{f_{j_k}\}$  that converges to the function  $g$  uniformly. We have, by Morera's theorem that  $g \in H(\mathbb{D})$ . If  $g(\mathbb{D})$  contains a boundary point of  $\mathbb{D}$ ,  $g$  turns out to be a constant function by Open mapping theorem. Otherwise,  $g(\mathbb{D}) \subseteq \mathbb{D}$  and hence  $g \in F$ . This clearly shows that  $F$  is a normal family and  $\mathbb{D}$  is a *taut*.

**Non-example:**

The domain  $\Omega \setminus \mathbb{C} \setminus \{0\}$  is not *taut*. If we define a sequence of holomorphic functions from  $\mathbb{D}$  to  $\Omega$  by  $f_n(z) = e^{nz}$  then  $f_n(0) = 1$  for every  $n$  but  $f_n(1/2) = e^{n/2} \rightarrow \infty$  as  $n \rightarrow \infty$ .

So, we don't have a convergent subsequence or a compactly divergent subsequence and hence  $\Omega$  is not *taut*.

Theorem 5.6.

*A domain  $\Omega \subseteq \mathbb{C}$  is hyperbolic if and only if it is taut.*

*Proof:*

$\Rightarrow$  Suppose that  $\Omega$  is hyperbolic, then the note following theorem 3.22 implies that  $\mathbb{C} \setminus \Omega$  should contain at least two points. Then the theorem 2.16(Montel) shows that  $\Omega$  is *taut*.

$\Leftarrow$  Assume that  $\Omega$  is *taut*. Then  $\mathbb{C} \setminus \Omega$  should contain at least two points otherwise we get a contradiction from above non-example of *taut*. Again by the same note following theorem 3.12, we conclude that  $\Omega$  is hyperbolic.

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## **An Amazing Legacy: a brief survey of the collected works of Prof. V.Lakshmikantham**

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Dr. V.Lakshmikantham (1926-2012) was an internationally well-known mathematician. He was professor and head of the Department of Mathematical Sciences at Florida Tech, Melbourne, Florida, USA. He was the founder (1992) and was the president (till 2010) of the "International Federation of Nonlinear Analysts". He organized several International conferences during his tenure at the University of Texas, at Arlington, Texas from 1973-1987. He organized five "World Congress of Nonlinear Analysts" conferences held in Tampa, Florida, (1992), Athens, Greece (1996), Catania, Italy, (1996), and Orlando, Florida (2004 and 2008) respectively. He is responsible, as a researcher, for initiating, publicizing, and popularizing several areas of Nonlinear Analysis. This is evident by his numerous research monographs and over five hundred published papers. He was the editor of five international mathematical journals and served on many editorial boards of journals. Apart from his research activity and teaching obligations and administrative demands, he still found time to write "The Origins of Human Past", (Children of the Immortal Bliss) a monograph published in 1980. This has also been translated into Malayalam, one of the south Indian languages. His final research monograph "Theory of Differential Equations in Cones" was published in 2011. He was a remarkable friend, philosopher and mentor to many in the Mathematical community.

Dr. V.Lakshmikantham's early education experience has been one of sheer personal grit and determination to overcome adversity and forge out a path that has kept him on this long road of purposeful career and scholarship. Knowing that, in 1956, he started his educational pursuit to work for PhD without a supervisor and in an institution where the atmosphere of creative scholarship was not existing, it is truly wonderful to see him at the acme of achievement after 50+ years down the road. In spite of the many curveballs he had to experience along the way, his life of purpose and achievement is indeed an inspiring story.

### **1. Introduction**

Professor V.Lakshmikantham's outstanding scientific work in the field of mathematics is quite awe-inspiring for its immensity, diversity and significance. His major contributions to several areas of nonlinear analysis and their lasting profound influence on the establishment of many important research areas speak to the significance of his work. His work is also significant for the vast extent and variety of problems he addressed, the introduction of important unifying principles and powerful techniques which have become an integral part of the methodology used in problems of nonlinear analysis. The following list provides a brief description of some seminal ideas and in later sections, we shall deal with these principles in more detail.

1. Comparison principle (a function satisfying a certain differential inequality is majorized by solution of a simpler comparison equation);
2. Comparison method in terms of Lyapunov-like functions (giving extraordinary scope and advantage in the qualitative study of nonlinear problems);
3. Extension and refinements of Lyapunov theory of stability (enhancing the applicability of that useful method in the analysis of large-scale interconnected systems of Science, Engineering, Economics and Biomathematics);
4. An effective and constructive mechanism to obtain upper and lower bounding monotone sequences (in which the method of upper and lower solutions combined with monotone iterative technique and method of quasi-linearization have played an important role in unifying a variety of nonlinear problems in continuous and discontinuous systems);
5. Many new results of stability criteria such as eventual stability, conditional stability, total stability etc., and several kinds of invariant sets and their analysis;
6. Stability in terms of two measures (a notion that unifies various kinds of stability of systems in a single framework, highlighting the essential ideas in such a study);
7. Method of vector Lyapunov functions and the theory of cone-valued Lyapunov functions;

8. Variational Lyapunov method to study perturbed systems by combining the method of variation of parameters and the Lyapunov method.

Some new areas of research that have evolved out of his work are

- i. dynamic systems on time scales (incorporating both continuous and discrete systems) [31];
- ii. systems with impulse effect (natural description of observed phenomenon in several real world problems) [20];
- iii. hybrid systems that are models for networks of digital and continuous devices (in which continuous and discrete-time processes governed by differential and difference equations are studied).

The other types of differential systems that are studied at length by Professor Lakshmikantham are

- 1) systems with finite and infinite delay [2, 30];
- 2) differential equations in abstract spaces [3, 5, 7];
- 3) fuzzy differential equations and inclusions [40];
- 4) set differential equations in metric spaces [42];
- 5) stochastic differential equations [6];
- 6) difference equations [17, 38];
- 7) integro-differential equations [29];
- 8) fractional differential equations [46];
- 9) monotone flows [39];
- 10) monotone iterative techniques [14, 27, 36];
- 11) nonlinear integral equations in abstract spaces [32];
- 12) causal dynamic systems [49];
- 13) differential equations in cones [50].

The numerous references in all the above cited monographs demonstrate the vast expanse of his research work in various branches of Nonlinear Analysis. Some of other areas of his research not included in the above cited references are (1) boundary value problems [4]; (2) oscillation theory of differential equations with deviating arguments [16] and (3) uniqueness and non-uniqueness criteria for ordinary differential equations [26].

In subsequent sections, we shall elaborate a little more on some specific topics that are investigated by Prof. Lakshmikantham and his vast group of followers, who have experienced his mentoring and nurturing.

## 2. Differential inequalities and comparison principle

The start of his mathematical research career in 1956 was his keen observation that the method of integral inequalities (to obtain results such as uniqueness, convergence of successive approximations, boundedness and asymptotic behavior of solutions) demanded an assumption of monotony in the function involved and a great deal of improvement is achieved by dealing directly with differential inequalities where that monotony condition is no longer required. Let us see this simple idea and its evolution in detail.

Consider the differential system

$$x' = f(t, x), \quad x(t_0) \tag{2.1}$$

where  $f(t, x) \in C[R_+ \times R^n, R^n]$ ,  $\|f(t, x)\| \leq g(t, \|x\|)$ ,  $g \in C[R_+ \times R_+, R_+]$ .

Converting (2.1) to an integral equation and using the estimate on  $f$ , we get the integral inequality for ordinary differential equations [26],

$$m(t) \leq m(t_0) + \int_{t_0}^t g(s, m(s)) ds, \quad t \geq t_0 \tag{2.2}$$

where  $m(t) = ||x(t)||$ , and  $x(t)$  is any solution of (2.1). If  $g(t, u)$  is monotone nondecreasing, using the theory of integral inequalities, we get

$$m(t) \leq r(t, t_0, u_0), \quad t \geq t_0, \tag{2.3}$$

whenever  $m(t_0) \leq u_0$ , where  $r(t, t_0, u_0)$  is the maximal solution of the scalar equation

$$u' = g(t, u), \quad u(t_0) = u_0 \tag{2.4}$$

Instead of using (2.2), we can use, for sufficiently small  $h > 0$ , the inequality

$$|m(t+h) - m(t)| \leq \int_t^{t+h} g(s, m(s)) ds$$

from which we get the differential inequality,  $|m'(t)| \leq g(t, m(t))$ , a.e and consequently the estimate (2.3) without requiring the nondecreasing nature of  $g$ . See [13'] for details. It is even enough to have

$$Dm(t) \leq g(t, m(t)) \tag{2.5}$$

where  $Dm(t)$  any one of the Dini derivatives. This observation implies that all we need is to require

$$||x + hf(t, x)|| \leq ||x|| + hg(t, ||x||) + o(h) \tag{2.6}$$

in place of  $||f(t, x)|| \leq g(t, ||x||)$ . The advantage of (2.6) is that  $g$  need not be nonnegative either and thus, we get a better estimate for  $||x(t)||$ . See reference [12']. This small idea grew into the famous comparison principle with  $V(t, x)$  a nonnegative function, with some properties of norm satisfying

$$V(t+h, x + hf(t, x)) \leq V(t, x) + hg(t, V(t, x)) + o(h)$$

where  $V \in C[R_+ \times R^n, R_+]$  and is locally Lipschitzian in  $x$ . Setting  $m(t) = V(t, x(t))$ , we get the differential inequality  $D_+V(t, x) \leq g(t, V(t, x))$ ,  $(t_0, x_0) \leq u_0$ , from which the estimate  $V(t, x(t)) \leq r(t, t_0, u_0)$ ,  $t \geq t_0$  is obtained. Thus we see that the comparison principle is an inherent consequence of the theory of differential inequalities, with a Lyapunov-like function serving as a versatile measure instead of  $||x||$ .

In the initial papers of V.Lakshmikantham starting with 1957, we see the systematic development of the theory of differential and integral inequalities employing the norm and Lyapunov-like functions, to obtain several results such as uniqueness and non-uniqueness, convergence of successive approximations, stability and boundedness, during 1957-1962. The comparison method in terms of Lyapunov-like function and appropriate comparison functions is the central theme in all generalizations, extensions and refinements of Lyapunov's second method, carried out during 1962-1967. During this time, several new concepts in stability theory such as eventual stability, conditional stability were introduced. It was recognized that the notions of Lyapunov stability and boundedness of different kinds of invariant sets such as asymptotically self-invariant sets, conditionally invariant sets, shift invariant sets etc., lead to weaker concepts of stability and boundedness. The advantage of using vector Lyapunov functions, introduced by Bellman and Matrosov, was recognized and utilized successfully to investigate and refine the Lyapunov theory of stability. Extensions of these ideas to differential equations with finite and infinite delay, differential equations in a Banach space, parabolic differential equations, complex differential equations, integro-differential equations and control systems were carried out. Construction of Lyapunov functions and converse theorems were also investigated.

The intense activity of Lakshmikantham and his collaborators during the period of 1964-1969 culminated in the publication of the famous two volume monograph, *Differential and Integral Inequalities*, published by Academic Press in 1969. See [1, 2] for details. Volume I is a book on the theory of differential equations and Volterra integral equations, containing for the first time a chapter on the method of vector Lyapunov functions. This inspired a lot of work on the large scale dynamic systems. Also see [13', 14'].

The second volume contains delay differential equations, partial differential equations of first order, parabolic and hyperbolic types, differential equations in a Banach space, nonlinear evolution equations and complex differential equations. The extension of Lyapunov stability theory in terms of Lyapunov functions and functionals and the basic theory of Razumikhin's method in terms of differential inequalities are considered in detail. With respect to the study of partial differential inequalities, a variety of comparison results are discussed. A priori bounds of solutions of partial differential equations in terms of solution of ordinary differential equation as well as solution of auxiliary partial differential equation are obtained. Lyapunov like functions are used to derive sufficient conditions for stability criteria. Relative to partial differential equations of parabolic type, basic results of parabolic differential inequalities (both in bounded and unbounded domains) are obtained. Lyapunov functions (scalar as well as vector) together with basic comparison theorems are employed to study stability and boundedness of solutions.

Though similar to Szarski's [13'] and Walter's [14'] books with similar titles, the treatment in [1, 2] is much more comprehensive, providing an excellent compendium of indispensable tools for researchers in this field. The two volumes showcase the fundamental role played by differential and integral inequalities together with the use of Lyapunov-like functions in establishing qualitative properties for a variety of problems such as existence, global existence, uniqueness, asymptotic behavior and stability of solutions of nonlinear equations. It is not possible to list out the vast number of results that are obtained in these two monographs and it is heartening to see that even after four decades, these books are still being used.

### 3. Stability analysis

The fruitful technique of Lyapunov's second method gained increasing significance and gave decisive impetus for modern development of stability theory. It was widely recognized that Lyapunov-like functions serve as a vehicle to transform a given complicated dynamic system into a relatively simpler system (a scalar comparison equation) and under appropriate conditions; the properties of the simpler system imply the corresponding properties for the original system. There are many concepts of stability like eventual stability, conditional stability, partial stability, relative stability and total stability to name a few. In order to unify the various kinds of stability, a notion of "stability in terms of two measures" was introduced, which has become a versatile method to study various kinds of stability of systems in a single frame work. The interplay of two different measures (one for the initial state and another for the state of the solution at time  $t$ ) offers new insight into the stability problem, paving the way for new applications [25].

The notion of practical stability is introduced to cover a situation where the system, though not stable in the sense of Lyapunov, has its deviations from the equilibrium state within certain pre-assigned bounds determined by the physical conditions. The notion of practical stability which is neither weaker nor stronger than Lyapunov stability has turned out to be a useful concept for many practical reasons and is investigated in detail in [23].

The method of vector Lyapunov functions has proved to be a very efficient tool in theory and applications for large scale interconnected dynamic systems occurring in diverse areas of real world problems. However, it is observed that the one drawback of this method is that it

requires the comparison system (of lower dimension than the original) to be quasi-monotone non-decreasing. In case of linear systems, this means that the non-diagonal elements of the comparison matrix to be nonnegative and quasi-monotonicity of the matrix is not a necessary condition for the matrix to be stable. In order to overcome this conflict and disadvantage, Lakshmikantham came up with the idea of choosing an appropriate cone other than  $R_+^n$  in which the partial ordering can be considered. This led to the development of cone valued Lyapunov functions [22, 15’].

Another approach to deal with the lack of quasi-monotonicity in the method of vector Lyapunov functions is to use the concept of quasi-solutions of the comparison system. The use of coupled quasi-solutions together with cone valued Lyapunov functions offers greater flexibility in studying large scale systems. These ideas together with the “decomposition and aggregation” method of Siljak [15’] will prove to be useful in the study of a broad class of interconnected systems. The monographs [19, 21-25] contain an extensive discussion of all the above mentioned ideas and approaches, which have contributed enormously to the development of stability analysis. Several ideas need further investigation to fully realize their applicability.

#### 4. Variational Lyapunov method

The two important tools that are available to study perturbed systems are comparison principle via Lyapunov-like functions and the method of variation of parameters (MVP). For a long time it was believed that MVP is applicable only to nonlinear problems with linear parts giving rise to linearization techniques. Alekseev’s work has made it possible to use the nonlinear variation of parameters formula to study nonlinear systems that have no linear parts, but have smooth nonlinear parts. An important idea of V.Lakshmikantham has been to combine the method of variation of parameters and the comparison principle via Lyapunov-like functions in a very effective way so that it provides a flexible mechanism to preserve the useful nature of the perturbations. This is done by way of a comparison theorem that connects the solutions of perturbed and unperturbed differential systems in terms of the solution of a suitable comparison equation. This method is now known as “Variational Lyapunov method” and is extensively used to study perturbed systems.

A typical comparison result in this approach is to get the estimate

$$V(t, x(t, t_0, x_0)) \leq r(t, t_0, u_0), \quad t \geq t_0$$

provided  $V(t_0, y(t_0, x_0)) \leq u_0$ , where

- (i)  $x(t, t_0, x_0)$  is any solution of the perturbed equation  $x' = F(t, x)$ ,  $x(t_0) = x_0$  and  $(t, t_0, u_0)$  is the maximal solution of the comparison equation  $u' = g(t, u)$ ,  $u(t_0) = u_0 \geq 0$  and  $y(t) = y(t, t_0, x_0)$  is any solution of the unperturbed equation  $y' = f(t, y)$ ,  $y(t_0) = x_0$ ;
- (ii)  $F(t, x) = f(t, x) + R(t, x)$ ,  $R(t, x)$  being the perturbation, with appropriate estimates;
- (iii)  $V(t, x)$  is the Lyapunov function, locally Lipschitzian in  $x$  and satisfying the differential inequality

$$D^+V(s, y(t, s, x)) \leq g(s, V(s, y(t, s, x))),$$

$D^+V$  being the Dini derivative of  $V$ , which is defined by

$$D^+V(s, y(t, s, x)) = \limsup_{h \rightarrow 0^+} \frac{1}{h} [V(s + h, y(t, s + h, x + hF(s, x))) - V(s, y(t, s, x))].$$

This comparison result [8’] is a natural extension and allows all special cases of the general stability theory and very useful in considering perturbed systems. It is interesting to note that in this case, under appropriate conditions, we can get the perturbed system to be uniformly asymptotically stable even though the unperturbed system is only uniformly stable. In [35],

we find a detailed extension and generalization of MVP to a variety of dynamic systems, continuous as well as discrete. Also, the blending of the two important techniques, namely MVP and MLF (method of Lyapunov-like functions and comparison principle) has been shown to be a unified front to deal with differential systems.

After introducing main ideas that are involved in the development of the linear and nonlinear variation of parameters formulae, several typical results concerning qualitative properties are provided as applications of MVP to

- (i) ordinary differential equations;
- (ii) integro differential equations (showing the difficulties involved in direct extension);
- (iii) delay differential equations (including neutral differential equations);
- (iv) differential equations in a Banach space (using semi-group approach);
- (v) impulsive differential equations and
- (vi) stochastic differential equations.

Some other equations that are considered in this context are differential equations with piecewise constant delay, dynamic systems on measure chains and hyperbolic differential equations. MVP for matrix differential equations is discussed bringing out the usefulness of Kronecker product of matrices. The generalization of variation of parameters formula in terms of Lyapunov-like function has been carried out in the setting of comparing the solutions of two different differential systems so that the perturbed differential system becomes a special case.

### **5. Systems with finite and infinite delay**

Systems with delay include differential equations and integro-differential equations with finite and infinite (or unbounded) delay. The development of the theory of functional differential equations with finite delay has progressed dramatically and several books dealing with this branch of research are available. Lakshmikantham in [2] has given various comparison theorems that are appropriate for delay differential equations and has extended Lyapunov theory to such equations. In extending the second method of Lyapunov to differential equations with time lag, one has a choice of treating the solutions as elements of a function space or as elements of the Euclidean space for all future time except at initial time and also a choice of employing Lyapunov functionals or functions.

Using Lyapunov functionals demands prior knowledge of solutions of the equation under consideration. On the other hand, in Razumikhin's method, employing Lyapunov functions depends crucially on choosing appropriate minimal sets of a suitable space, along which the derivative of Lyapunov function admits an appropriate estimate. If the estimation of the derivative of Lyapunov is done by means of a functional, the selection of minimal sets is not necessary. The notion of maximal solution for functional differential equations and theory of functional differential inequalities are required to apply the comparison principle analogous to the use of comparison principle in ordinary differential equations. It is important to note that the knowledge of solutions is not required in either case.

However, when one examines these two approaches adopted by many researchers, we find that Lyapunov functionals constructed in examples are indeed a combination of a Lyapunov function and a functional in such a way that the corresponding derivative can be estimated suitably without demanding prior knowledge of solutions or the minimal sets. This observation has led to the method of Lyapunov functions on product spaces which serves well for all types of equations, with bounded or unbounded delays. Initially, the theory of equations with unbounded delay was not fully developed but the survey paper of Lakshmikantham and Corduneanu [2'] provided the needed impetus for the development of that theory. See also, [16'].

Most results in the oscillation theory of delay differential equations started with extending the corresponding results for equations without delay. However, the work by Lakshmikantham and associates that show delay causes oscillations generated a lot of interest. The monographs [16, 29, 30] have influenced further work in the area of delay equations.

### **6. Boundary value problems**

In the early seventies, Lakshmikantham's attention was focused on the application of differential inequalities and Lyapunov-like functions to the theory of nonlinear boundary value problems. Due to the entirely different nature of the underlying process, its study is more challenging than that of initial value problems. The substantial results that were obtained in that study can be identified as

- (i) the use of vector of Lyapunov-like functions and the theory of differential inequalities is established as a useful method to study boundary value problems on finite or infinite intervals as well as boundary value problems of functional differential equations;
- (ii) establishing that whatever is achieved by the application of Leray-Schauder's alternative can also be realized by the modified function technique under the same set of conditions, thus shedding more light on apparently diverse approaches to study boundary value problems;
- (iii) the use of angular function technique together with Lyapunov-like functions and differential inequalities to obtain existence of solutions for nonlinear boundary value problems with generalized boundary conditions. The monograph [4] contains much updated information and a detailed discussion of various techniques that are employed in the study of boundary value problems and covers the contribution of the Russian school which was not well-known in the English speaking world.

### **7. Differential equations in abstract spaces**

Lakshmikantham's initial interest in differential equations in Banach spaces started with the extension of the theory of differential inequalities and Lyapunov's method to such equations in which certain conditions are imposed on the resolvent operator in terms of a Lyapunov function to obtain results such as global existence, uniqueness, bounds on norms of solutions and approximate solutions, uniqueness, stability and boundedness of perturbed systems generalizing the results of Mlak and Kato. The work continued in this direction and fundamental results such as uniqueness, continuous dependence and differentiability of solutions with respect to initial data, nonlinear variation of parameters formula and asymptotic equilibrium were established. The significant results concerning lower bounds and uniqueness of solutions of evolution inequalities in a Hilbert space, generalizing the works of Agmon, Nirenberg and Ogawa, were obtained and these provided a wide range of applicability to partial differential equations.

Foreseeing the importance of this fascinating area of research and equipped with the theory of linear evolution equations developed by Sobolevskii and Tanabe, Lakshmikantham and Ladas wrote the monograph [3]. This is a self-contained text that appeals to the beginner as well as the specialist. In [3], a comprehensive treatment of the required functional analysis background is given together with the calculus of abstract functions (functions mapping an interval of real line into a Banach space), theory of semi-groups of operators which are needed for the study of differential equations in Banach spaces. Other topics that are discussed are linear evolution equations of parabolic type (Sobolevskii-Tanabe theory), evolution inequalities, nonlinear differential equations and several special topics such as nonlinear semi-groups, functional differential equations in Banach spaces and 2<sup>nd</sup> order evolution equations are considered. As a natural consequence, this monograph generated a lot of interest and provided a great impetus for furthering the theory of nonlinear equations and

nonlinear semi-groups. Lyapunov's method was extended to obtain several stability criteria of nonlinear evolution equations.

In [7], we find a presentation of the significant development of the theory of nonlinear ODEs in an arbitrary Banach space during 1965-1980. It contains the existence theory for differential equations with continuous and discontinuous right hand sides, fundamental properties of solutions, qualitative theory and some special topics such as abstract boundary value problems, monotone iterative methods, delay differential equations. Also notions such as duality mappings, generalized inner products are introduced. Measures of non-compactness relative to strong and weak topologies are discussed and a variety of comparison results are given. Some important areas of investigation that are included in this monograph are the existence results in closed sets under dissipative as well as compactness type conditions with a needed boundary condition. The study of Cauchy problem for differential equations in a Banach space has been done in two ways:

- (i) using compactness type conditions to guarantee existence of solutions (similar to classical Peano theorem);
- (ii) using dissipative type condition that assures existence and uniqueness of solutions (similar to classical Picard theorem).

The lecture notes [5] deals with stability and asymptotic behavior of solutions and this generated further interest in the study of qualitative behavior of solutions, supported by theory of cones, the notion of measure of non-compactness and existence results in closed sets. The study of evolution equations with set-valued discontinuous right hand sides resulted in fundamental works of Crandall and Liggett, Kobayashi and Benilan. This important state of the art in nonlinear evolution equations formed the monograph [7] which together with the books of Barbu, Deimling and Martin gives up-to-date information on the theory of evolution equations.

### **8. Impulsive differential equations**

Many evolution processes describing real world phenomena are characterized by the fact that at certain moments of time, the system experiences a sudden change of state. This abrupt change in state is due to the short term perturbations whose duration is negligible in comparison with the duration of the process. It is therefore natural to treat such perturbations as impulses and study impulsive differential equations. These impulses can occur at fixed times or at variable times (time-dependent). The theory of differential equations with impulses is much richer than the corresponding theory of differential equations without impulse effect. The monograph [20] generated a lot of interest and is responsible to make this branch of research so popular.

In [20], the evolution processes subject to impulsive effects are discussed in detail and their qualitative properties are studied. Typically, an impulsive differential system consists of three parts:

- (i) a system of differential equations which describe the evolution of the state at times  $t \neq t_k$ ;
- (ii) the moments of impulsive effect  $t = t_k, k = 1, 2, \dots$  which are either fixed moments of time or moments depending on the state of the solution,  $t = \tau_k(x)$  and
- (iii) the change experienced by the solution at those moments of impulse effect given by  $\Delta x = I_k(x), t \neq t_k$  where for  $t = t_k, \Delta x(t_k) = x(t_k^+) - x(t_k)$  and  $x(t_k^+) = \lim_{h \rightarrow 0^+} x(t_k + h)$ .

Even in the simplest case of impulses at fixed moments of time, the behavior of solutions is influenced by the type of impulsive effect. Systems with variable moments of impulse effect offer more difficult and challenging problems. The solutions starting at different points will

have different points of discontinuity and strange behaviors like “pulse phenomenon” and “confluence” are exhibited by solutions of such systems.

However, all such systems are studied in [20] under appropriate conditions and the theory of impulsive differential inequalities together with extension of Lyapunov’s method are developed for such equations. Important features of this monograph are

- (i) impulsive integral inequalities;
- (ii) fundamental properties of solutions such as existence, uniqueness, continuous dependence on initial data;
- (iii) monotone iterative techniques;
- (iv) stability analysis via Lyapunov functions;
- (v) stability in terms of two measures;
- (vi) systems with variable structure and
- (vii) integro-differential systems with impulses.

In the last two decades we see a significant increase in the study of impulsive differential equations, which are suited for many applications. Much of the work deals with impulses at fixed moments of time. Many results were obtained by A.A.Martynyuk [11’] via matrix-valued Lyapunov functions. Due to the inherent difficulties which require new ideas to overcome, the study of impulsive equations with impulses at variable moments (depending on the state of the solution) has not developed satisfactorily.

### 9. Stochastic differential equations

The mathematical modeling of several real world problems leads to differential systems that involve some inherent randomness due to imprecise information or uncertainties. If the randomness is ignored or eliminated, we have deterministic systems. But, for the realistic modeling of many phenomena, we have to pay attention to the effect of randomness which can influence the differential system in three ways:

- (i) random initial conditions,
- (ii) random forcing functions and
- (iii) random coefficients.

Thus, the study of stochastic differential equations gained importance and is a challenging area or research. Since the solutions of such equations are stochastic processes, their qualitative study requires appropriate probabilistic modes of analyses. Strongly motivated and influenced by the idea that Lyapunov functions together with theory of differential inequalities will provide a universally effective tool, the theory of random differential inequalities and the concept of random Lyapunov functions were successfully developed to study random differential equations. These investigations resulted in the monograph [6] in which we find the following important features:

- (i) the study of random differential equations through suitable calculus, depending on the different modes of probabilistic analysis, namely, sample calculus,  $L^p$  – mean calculus and Ito-Doob calculus;
- (ii) development of the theory of random differential inequalities through the different modes of probabilistic analysis and application of these results to discuss various properties of solution processes;
- (iii) a unified treatment of stability theory through random Lyapunov functions and random comparison method and
- (iv) emphasis on the role of the method of variation of parameters in the stability analysis of stochastic perturbed systems.

## 10. Difference equations

It is well known that the study of discretization methods for differential equations leads to difference equations. Also, difference equations are natural descriptions of observed evolution of phenomena because most measurements of time dependent variables are discrete and as such these equations are important mathematical models. In [17], we find a systematic treatment of the theory of difference equations and its applications to numerical analysis. Several results in this theory have been obtained as more or less natural discrete analogues of corresponding results of ordinary differential equations, especially in the case of Lyapunov theory of stability. However, it is worth noting that the theory of difference equations is a lot richer than the corresponding theory of differential equations. For example, a simple difference equation resulting from a first order differential equation may exhibit a phenomenon called appearance of “ghost” solutions or existence of chaotic orbits that can happen only for higher order differential equations.

In [17], the applications of the theory of difference equations to numerical analysis, control theory, population dynamics, economics and queuing theory are described in detail. The monograph also features

- (i) a systematic development of the theory of difference inequalities and various comparison results and
- (ii) a unified treatment of stability theory through Lyapunov functions and comparison method. This monograph has gone into a second edition incorporating many improvements and published by Marcel Dekker [38].

## 11. Dynamic systems on time scales

Differential equations and difference equations have always played a prominent role in mathematical modeling of several important dynamic processes. From a modeling point of view, it is perhaps more realistic to model a phenomenon by a dynamic system which incorporates both continuous and discrete times, namely, time as an arbitrary closed set of real's called measure chain or time scales. It is therefore natural to ask whether both dynamic systems (difference equations and differential equations) can be handled in a single framework. The theory of dynamic systems on time scales, which is recently developed, offers such a unified approach. In [31], Lakshmikantham and associates have presented the current state of development of such systems. The necessary calculus of functions on time scales is developed following Aulbach and Hilger and the basic properties of such dynamic systems are investigated. Lyapunov stability theory is discussed in this context, utilizing appropriate dynamic inequalities and comparison results. Other areas of investigation that are carried out are

- (i) monotone iterative technique
- (ii) method of quasi-linearization
- (iii) monotone flows
- (iv) practical stability of large scale uncertain dynamic systems and
- (v) boundary value problems.

This monograph provides an impetus to investigate other important problems in this framework of time scales and it has already been the forerunner for a lot of work in this direction (see, for instance, [9']).

## 12. Nonlinear problems in abstract cones

In many problems that arise from models of chemical reactors, neutron transport, population dynamics, infectious diseases, economics and other systems, one needs to consider the

existence of nonnegative solutions with desired qualitative properties. The concept of nonnegative can be developed through arbitrary cones, which define the partial ordering in the space to be discussed.

The advantage of the theory of cone valued Lyapunov functions in stability analysis is well known. Employing the theory of cones coupled with fixed point index, one can discuss positive fixed points of nonlinear operators. These results can be utilized to investigate positive solutions of nonlinear differential and integral equations of Fredholm, Volterra and Hammerstein types. The two monographs [18, 32] deal with the investigation of nonlinear problems in abstract cones and provides several interesting applications to real world problems. The topics that are discussed in detail are

- (i) nonlinear integral equations of Fredholm, Volterra and Hammerstein types;
- (ii) first and second order nonlinear integro-differential equations in Banach spaces including equations of Volterra type and equations of mixed type;
- (iii) nonlinear impulsive differential equations in Banach spaces and their applications.

In order to illustrate the abstract results, several examples of both finite and infinite systems of scalar integral equations, differential equations, integro-differential equations and impulsive equations are given.

### **13. Integro-differential equations**

The origins of the study of integral and integro-differential equations may be traced back to the works of Abel, Lotka, Fredholm, Malthus, Verhulst and Volterra on problems in mechanics, mathematical biology and economics. The problem of competing species as studied by Volterra has contributed much to the development of mathematical modeling of many real world problems. Though Volterra integro-differential equations with bounded and unbounded delays have been investigated by many researchers over the past few decades, there were no books available, exclusively devoted to the theory of integro-differential equations as a subject in itself. The monograph [29] fills that void and provides the basic theory and qualitative properties of solutions of Volterra integro-differential equations and a large number of applications. The basic theory deals with existence, uniqueness, continuous dependence and differentiability of solutions with respect to initial data, integro-differential inequalities and comparison results. Also, linear and nonlinear variation of parameters, monotone iterative techniques are considered.

The monograph contains topics such as linear and weakly nonlinear systems together with fundamental properties of stability, boundedness and periodicity of solutions. Equations with impulse effects, and difference equations resulting from Volterra integro-differential equations, which are of great interest to engineers and applied scientists, are discussed. Lyapunov stability and its refinements are considered extensively together with construction of Lyapunov functions and functionals, Lyapunov-Razumikhin method and Lyapunov functions on product spaces. Stability analysis of nonlinear equations with impulse effects, engineering systems such as input-output systems and large scale systems is covered. Qualitative properties of solutions of integro-differential systems that arise in problems of biological population dynamics, grazing systems, wave propagation, nuclear reactors and visco-elasticity are investigated.

### **14. Uniqueness and non-uniqueness**

In the monograph [26], Lakshmikantham has addressed the question of dealing with the enormous amount of literature that exists offering various sufficient conditions for uniqueness problems, with implied conditions for existence of solutions. Though this age old problem of great importance has been studied, refined and generalized, there are still unanswered

questions. For example, existing uniqueness criteria cannot offer any answer to the pathological example  $x' = x^\alpha + t^\beta$ ,  $0 < \alpha < 1$  and  $\beta > 0$ . Lakshmikantham and associates have contributed several important results in this area. In [26], they have made an attempt to codify existing results with a view to enable the young researchers to compare and contrast various existing results and to explore further with new ideas to solve open problems.

### 15. Monotone iterative techniques

Generally speaking, the methods of proving existence results for nonlinear differential equations consists of three steps:

- (i) constructing a sequence of some kind of approximate solutions;
- (ii) showing the convergence of the constructed sequence of approximate solutions; and
- (iii) proving that the limit function is actually a solution of the given nonlinear problem.

Out of these three steps, it is step (ii) that deserves attention since the other two steps are fairly standard. The sequence of approximate solutions can be shown (i) to be a Cauchy sequence or (ii) that a uniformly convergent subsequence can be extracted with the application of Ascoli-Arzelà theorem or (iii) that it is a monotone sequence that converges uniformly. It is this third choice that gives us the method of monotone iterative technique, which yields constructive proofs of existence results and also suitable numerical procedures for computation of solutions. In [14], Lakshmikantham and associates have dealt extensively with the method of upper and lower solutions, coupled with monotone iterative technique and also have shown that it offers theoretical as well as constructive existence results in a sector. The upper and lower solutions that generate the sector serve as upper and lower bounds for solutions which are improved successively by monotone iterative procedure. In recent years, the procedure is shown to be of immense value and is unifying a wide variety of nonlinear problems.

A basic result in this context can be described as follows (in the simple scalar case):

If  $v_0$  and  $w_0$  are lower and upper solutions of the scalar initial value problem

$$u' = f(t, u), \quad u(0) = u_0 \quad (15.1)$$

(with  $v_0' \leq f(t, v_0)$ ,  $v_0(0) \leq u_0$  and  $w_0' \geq f(t, w_0)$ ,  $w_0(0) \geq u_0$ ) such that  $v_0 \leq w_0$  on  $J = [0, T]$  and  $f$  satisfies  $f(t, u) - f(t, \bar{u}) \geq -M(u - \bar{u})$  for  $v_0 \leq \bar{u} \leq u \leq w_0$ , and  $M > 0$ , then, there exist monotone sequences  $\{v_n\}, \{w_n\}$  such that  $v_n \rightarrow v$ ,  $w_n \rightarrow w$  uniformly and monotonically and that  $v, w$  are minimal and maximal solutions of (15.1) respectively.

For any  $\mu \in C[J, R]$  such that  $v_0 \leq \mu \leq w_0$ , the scalar equation

$$u' = f(t, \mu) - M(u - \mu), \quad u(0) = u_0 \quad (15.2)$$

which guarantees a unique solution for every such  $\mu$ , when we define a mapping  $A$  by  $A\mu = u$ . This will be used to define sequences  $\{v_n\}, \{w_n\}$  successively. The operator  $A$  is shown to be a monotone operator on the segment

$$[v_0, w_0] = \{u \in C[J, R]: v_0 \leq u \leq w_0\},$$

with the properties  $v_0 \leq Av_0$ ,  $w_0 \geq Aw_0$  and for  $\mu_1, \mu_2 \in [v_0, w_0]$ ,  $\mu_1 \leq \mu_2$ ,  $A\mu_1 \leq A\mu_2$ . This enables one to obtain the sequences  $\{v_n\}, \{w_n\}$  with

$$v_n = Av_{n-1}, \quad w_n = Aw_{n-1}, \quad n = 1, 2, \dots$$

Since each member of these sequences is the solution of the linear equation (15.2) explicitly computable, the advantage and importance of the method is clear.

In [14], this basic idea is exploited and applied to many other continuous nonlinear differential systems including first and second order differential equations, elliptic, parabolic and hyperbolic equations, both scalar case and finite systems. In each case, suitable relevant applications are discussed. In the monograph [36], this method of lower and upper solutions together with the method of quasi-linearization discussed by Bellman and Kalaba and the idea of Newton-Fourier, has been shown to be another useful way to construct concurrently lower and upper bounding monotone sequences which converge quadratically. This approach, now known as the method of generalized quasi-linearization, provides a unified mechanism and enlarges the class of nonlinear problems considerably to which the method is applicable.

The development of corresponding theory for discontinuous nonlinear problems has taken different directions. To investigate theoretical existence results, one can convert the given function which is discontinuous in the dependent variable into a set-valued map and then consider set-valued differential equations. Alternatively, one could employ Zorn's Lemma directly to prove existence results. However, the usual monotone iterative method is no longer applicable to prove constructive existence results. Owing to the intrinsic difficulties of relating the partial ordering with inclusion so as to reveal the monotone character of set-valued maps, no real progress is made in this direction. In spite of these difficulties, one can develop some generalized monotone iterative method that is suitable to deal with discontinuous nonlinear problems.

In [27], a generalized monotone iterative method in the context of partially ordered sets based on elementary set theory has been discussed, using well-ordered chains of iterations. It is then applied to derive suitable fixed point theorems in ordered abstract spaces, which are important tools in the systematic investigation of generalized monotone iterative method in terms of upper and lower solutions. The monograph also covers the following:

- (i) new existence and comparison results when the function involved in the differential equations admits a threefold decomposition, namely, continuous, Lipschitzian and discontinuous functions;
- (ii) the extensions to Caratheodory systems in finite and infinite dimensional spaces;
- (iii) a comprehensive treatment of existence and comparison of extremal strong, weak or mild solutions to discontinuous differential equations in ordered Banach spaces without needing any kind of compactness hypotheses.

The necessary basic analysis of ordered spaces, generalized iteration ideas and required fixed point theorems are included. Finally, in the monograph [39], these ideas are considered in the context of partial differential equations to study monotone flows and rapid convergence of the monotone sequences.

## **16. Fuzzy differential equations and inclusions**

We encounter two kinds of situations while modeling real world phenomena in terms of mathematical terms. One is the excessive complexity of the model, which beyond a certain threshold level, makes it difficult to make precise statements about its behavior and as a result we lose the significance of the model to represent the phenomenon realistically. The other kind of situation arises because of the indeterminacy and inability to differentiate and describe events in a real situation exactly. Natural language, with the vagueness of its semantics, is capable of dealing with vague notions, but classical mathematics cannot cope with such imprecise and vague notions. It therefore becomes necessary to have some mathematical apparatus to describe vague and uncertain notions and thereby help the modeling process of imprecise real world systems.

In 1965, Zadeh [3'] initiated the development of the modified set theory, known as fuzzy set theory, which serves as a tool to make the description of vague and uncertain notions

mathematically acceptable. The rise and development of new fields such as general system theory, robotics, artificial intelligence and language theory force us to be engaged in specifying imprecise notions. A fuzzy set is a function from a set into a lattice or as a special case, into the interval  $[0, 1]$ , with certain properties. Fuzzy set theory and its applications have been extensively developed since the 1970's and there are several books dealing with those investigations.

In [40], Lakshmikantham and associates have attempted to describe the recent developments of the fuzzy differential equations or inclusions and initiate several new extensions to other types of fuzzy differential systems. When a certain initial value problem is employed to represent a real world situation, we are not usually sure about the many uncertain parameters (that enter into the description of the underlying function) which are estimated through measurements subject to random errors. In this case, random differential equations with random data may be of help. But, if the underlying structure is not probabilistic, because of the variation of subjective choices, it would be natural to employ fuzzy differential equations. The first two chapters of the monograph contain necessary tools to deal with fuzzy set theory and the calculus of fuzzy functions. Chapter 3 focuses attention on the theory of fuzzy differential equations and brings out the intricacies involved in incorporating fuzziness into the theory of differential equations; Lyapunov-like functions and their role in studying stability criteria are discussed in chapter 4.

Though much of the work is still ongoing, the monograph makes an excellent introduction to the initial apparatus required for the study of fuzzy differential equations and introduces several new areas of investigation relative to fuzzy differential systems, providing few typical results and opens the way for further developments. One such opening is described in chapter 6 as the new and better framework of utilizing Hüllermeir's formulation of fuzzy IVP's instead of the earlier approach by Hukuhara.

### **17. Set differential equations in a metric space**

The study of analysis in metric spaces which has gained importance in recent times has led to the realization that many results of differential calculus and set valued analysis, including inverse function theorem, do not really depend on the linear structure and hence can be adapted to nonlinear case of metric spaces. Multivalued differential equations (now known as set differential equations (SDE) generated by multivalued differential inclusions) have been introduced in a semi-linear metric space, consisting of all nonempty, compact, convex subsets of an initial finite or infinite dimensional space. The basic existence and uniqueness results of such SDEs have been investigated and their solutions have compact, convex values. These generated SDEs have been employed to prove the existence of solutions of multi-valued differential inclusions and the multi-functions involved in this set up are compact, but not necessarily convex, subsets of the base space utilized.

Due to the recent developments in the theory of fuzzy differential equations, we are now aware of the disadvantage resulting from fuzzification of the derivative employed in the original formulation (Hukuhara) of fuzzy differential equations (FDE). An alternative formulation (Hüllermeir's approach based on a family of multivalued differential inclusions derived from the fuzzy maps involved in the FDE's) is suggested in [40] which results in SDEs reflecting the rich behavior of ordinary differential equations before fuzzification.

The investigation of the theory of SDEs as an independent discipline has certain advantages. For example, when the set is a single valued mapping, the Hukuhara derivative and the integral used in formulating SDEs reduce to the ordinary vector derivative and integral. The results obtained in the framework of such SDEs become the corresponding results of ODEs if the basic space is  $R^n$ . On the other hand, if the base space is a Banach space, we get from

such SDEs, the differential equations in a Banach space and we have only a semilinear metric space to work with in the SDE setup compared to the complete normed linear space that one employs in the study of ODEs.

In the monograph [42], Lakshmikantham has given a systematic account of relevant recent developments, describing the current state of the theory of SDEs and showing the essential unity achieved. Several new extensions to other types of SDEs are initiated. The monograph contains

- (i) the necessary preliminary material and calculus for set valued maps;
- (ii) the fundamental theory of SDEs such as comparison principles, existence, uniqueness and continuous dependence of solutions, existence of extremal solutions with suitable introduction of partial order in the metric space;
- (iii) monotone iterative technique using lower and upper solutions under the continuity assumption for SDEs. Also, method of non-smooth analysis, existence and flow invariance results without any continuity assumption are discussed in terms of Euler solutions.

Lyapunov theory is extended to SDEs, with suitable comparison results in terms of Lyapunov-like functions as in [10']. Stability and boundedness criteria are obtained by choosing appropriate initial values in terms of the Hukuhara difference to eliminate the undesirable part of the solutions of SDEs so that the rich behavior of the corresponding ODEs is preserved. Methods of vector Lyapunov functions and perturbing Lyapunov functions are discussed in detail. Stability results are obtained under weaker assumptions by employing lower semi-continuous Lyapunov functions and non-smooth analysis.

An important feature of this monograph is the interconnection between SDEs and FDEs. Necessary tools are provided for formulating FDEs and basic results are proved. The interconnection between FDEs and SDEs is explored via a sequence of multivalued differential inclusions, suitable generating SDEs. Impulsive effects are incorporated in FDEs and it is shown how impulses can help to improve the qualitative behavior of FDEs. Hybrid fuzzy differential equations are introduced and their stability properties are discussed. Several new topics such as impulsive SDEs, SDEs with time delay, set difference equations and SDEs involving causal maps which cover several types of SDEs including integro-differential equations, are initiated. Some important basic results are provided for each type of SDEs. The discussion of Lyapunov theory is carried out in terms of single, vector, matrix (see [11'] and references therein) and cone-valued Lyapunov-like functions by an appropriate choice of the metric space.

### 18. Causal dynamic systems

In the monograph [49], the theory of causal differential equations (or differential equations with causal operators, which are non-anticipative or abstract Volterra operators) is given in a comprehensive way. The CDEs (Causal Differential Equations) include a variety of dynamic systems and consequently the theory of CDEs brings out the unification achieved. Many tools which are employed for ODEs are applicable to CDEs, including the method of Lyapunov functions for stability theory and other qualitative properties of solutions of CDEs. The first book dealing with functional equations with causal operators is by Corduneanu [1'] and it has paved the way for a lot of further research in this area. The monograph [49] provides a systematic account of recent developments, describes the present state of the useful theory and initiates several new extensions to other types of CDEs such as CDEs in infinite dimensional spaces with or without memory, fractional CDEs and set differential equations with causal operators.

Let  $E = E(J, X)$ , where  $J$  is an appropriate time interval,  $X$  is a finite or infinite dimensional space (depending on the requirement of the context) so that  $E$  is a function space. The

operator  $Q: E \rightarrow E$  is said to be a causal operator if for each couple of elements  $x, y$  of  $E$  such that  $x(s) = y(s)$  for  $0 \leq t_0 \leq s \leq t$ , we also have  $(Qx)(s) = (Qy)(s)$  for  $0 \leq t_0 \leq s \leq t$ ,  $t < T$ ,  $T$  being arbitrary.

Corduneanu's work [1'] deals with the study of causal operators in great detail and in the very beginning, it provides examples of several dynamic equations that can be included in causal differential equations of the form  $x'(t) = (Qx)(t)$ ,  $x(t_0) = x_0$ . We can name a few dynamical systems that are described by this CDE setup:

- (i) IVP for ODEs;
- (ii) (ii) IVP for difference-differential equations;
- (iii) (iii) functional differential equations;
- (iv) (iv) integro-differential equations with finite memory;
- (v) (v) very general integro-differential equation with memory or delay incorporated.

In the monograph [49], after providing some necessary preliminary material and useful fixed point theorems, concepts of measures of non-compactness and non-convexity, Lakshmikantham and associates have investigated the fundamental theory of CDEs such as comparison principles, existence, uniqueness and continuous dependence of solutions on initial data, extremal solutions, global existence and Nagumo-type uniqueness. Existence of solutions via non-smooth analysis is discussed. The method of lower and upper solutions including monotone iterative technique, quasi-linearization and their generalizations is described in detail, providing a rich source of methods of finding approximate sequences which are monotone and converge to the extremal solutions/unique solution of the causal differential system. Lyapunov method is extended to CDEs by employing Lyapunov functions, Lyapunov functionals and functions on product spaces. Stability theory in terms of two measures provides the unification for the study of many different stability concepts relative to CDEs. The method of vector Lyapunov functions and cone valued Lyapunov functions are given.

Many new topics are initiated in the context of CDEs:

- (i) CDEs in a Banach space;
- (ii) CDEs with fractional derivatives;
- (iii) CDEs with memory;
- (iv) CDEs with retardation and anticipation;
- (v) causal set differential equations.

Few basic results are given in each case, enough to provide initial apparatus for further study. It is the first book that attempts to describe the theory of CDEs as an independent discipline, incorporating the recent general theory as well as the interconnection between various dynamic systems and CDEs.

## 19. Fractional dynamic systems

Although the concept of non-integer order derivative, known as fractional derivative, goes back to 17<sup>th</sup> century and the necessary calculus of fractional derivatives and integrals of arbitrary order was completed by end of 19<sup>th</sup> century, it is only a few decades ago it was realized that the derivatives of arbitrary order provide an excellent framework for modeling real world problems in a variety of disciplines. The first book of Oldham and Spanier systematically presents the applications from physics, chemistry and engineering. However, the theory of nonlinear fractional dynamic systems has not been investigated except for some existence and uniqueness results using contraction mapping principle. There has been a renewed interest in that theory in recent times, and with sufficient literature to warrant a concerted effort to assemble all existing fundamental results in a unified way,

Lakshmikantham's timely monograph [46] provides a systematic account of recent developments, describes the current state of the theory and initiate several new extensions to other types of fractional dynamical systems.

In the first chapter, after giving a short history of the subject and the needed preliminaries of Gamma function, and Mittag-Leffler function, the three kinds of fractional derivatives are introduced, namely, Riemann- Liouville (R – L), Grünwald-Letnikov (G – L) and Caputo. In chapter 2, results concerning fractional differential inequalities are established and compared with the corresponding differential inequalities of integer order. Some local existence and uniqueness theorems, existence of extremal solutions and necessary comparison results are provided. The solution of non-homogeneous linear fractional differential equation is obtained using the method of successive approximations since the method of variation of parameters cannot be utilized in the fractional case. Appropriate comparison results are obtained to consider finite systems of fractional differential equations, in order to extend the method of vector Lyapunov functions for the investigation of large scale fractional differential systems, with or without multi-order. Also, the initial value problem of Caputo type fractional differential equation is considered and some basic results of existence and uniqueness of local solutions, and continuous dependence relative to initial data are proved.

Chapter 3 deals with the extension of monotone iterative technique to fractional differential equations to study initial value problems and periodic boundary value problems. The relationship between those 3 types of fractional derivatives is discussed and some results are obtained for the estimation of solutions of fractional differential equation of R-L type in terms of solutions of differential equations of integer order, which is useful when multi-order fractional differential systems are considered. In chapter 4, Lyapunov stability theory is extended to fractional differential systems (of G-L type) via single and vector valued Lyapunov functions. After proving the corresponding two original theorems of Lyapunov in this framework, stability and boundedness criteria are discussed in terms of two measures, thus obtaining stability theory for fractional differential equations in a comprehensive way. Multi-order differential systems are considered and a technique is developed by which their study can be reduced to the study of ordinary differential systems. The idea involved in this development has great potential for further research and its applications.

Finally, in chapter 5, various new topics are introduced to enlarge the scope of investigation to other types of fractional differential equations such as equations

- (i) with past memory
- (ii) involving causal operators and
- (iii) in a Banach space.

Boundary value problems are considered in the context of fractional differential inclusions. Fractional evolution equations are studied for which existence of automorphic solutions is proved. For all these different types of differential systems, some typical results are proved in the framework of Riemann-Liouville fractional derivative. This is the first book that attempts to describe the theory of fractional differential equations as an independent discipline and provides enough material for sustained further advances of the subject.

## **20. Differential equations in cones**

The interest in obtaining the existence of nonnegative solutions with certain desired qualitative properties can be interpreted as the need to get existence results in arbitrary cones, which are closed, convex subsets of appropriate space under consideration. The cones define an order relation by means of which different elements can be compared in a way better than using estimates in terms of a norm. Also, the method of vector Lyapunov functions that is

needed for the study of large scale systems demonstrates that employing arbitrary cones yields better results compared to using component-wise inequalities.

Many problems in partial differential equations (PDEs) which arise from physical models have the form of evolution equations which can often be considered as ordinary differential equations (ODEs) in suitable infinite dimensional spaces. By imbedding PDEs in general evolution equations renders it possible to make use of the elegant, well developed theories and powerful techniques of ODEs in arbitrary Banach space and nonlinear functional analysis.

The monograph [50] deals with the theory of differential equations in cones and demonstrates that the development of the Lyapunov method in cones enhances the power of that method for the study of large scale systems. Chapter 1 provides the necessary preliminary material. In chapter 2, existence theory of local, global and extremal solutions in closed convex sets is considered along with flow invariance. Since cones are special cases of closed convex sets, the results provide the basic existence theory in cones. In chapter 3, monotone iterative technique and the method of quasi-linearization are discussed in the context of cones, thus giving an abstract version of those powerful approximation methods. Cone-valued Lyapunov functions and stability theory in terms of two measures are considered in chapter 4, providing many typical results and a comprehensive treatment of the Lyapunov method in cones. In order to facilitate further progress in this area of research, extensions to various other dynamic systems are indicated in chapter 5.

### **21. The Hybrid Grand Unified Theory**

In the recent work of Lakshmikantham with E.Escultura and S.Leela titled The Hybrid Grand Unified Theory [47], an attempt is made to provide a hybrid grand unified theory to understand the universe, both in its micro/quantum aspects as well as macro/galactic aspects, trying to encompass both modern and ancient theories of the universe, together with its functioning at all levels of human experience. In this attempt, it became necessary to acknowledge the ambiguity and limitations of mathematics relative to the fundamental concepts of very large numbers and very small numbers, infinity and the limiting process in general and the inadequacy of the mathematical modeling, consequent computation and measurement, to provide answers to the many unanswered questions in physics.

The monograph [47] brings out the appropriateness of the decimal system representation as a computational tool and the real number system is put on solid foundation without inconsistency, incorporating the notions of dark numbers and their duals, personal and impersonal infinity. The dark matter of physics and dark numbers of mathematics are correlated and the superstring is introduced with its nested fractal structure as the basic constituent of matter and the Cauchy representation of dark number is its mathematical model. The important features of the book, in addition to the above mentioned are:

(1) the hybrid approach of combining qualitative mathematics and computation to discover the natural laws in order to explain in a unified way several natural phenomena at micro/quantum and macro/galactic levels;

(2) matters of mind and consciousness, evolution and involution, creation and dissolution are considered to show the common hybrid interaction between modern and ancient scientific theories of nature and search for higher reality.

### **Epilogue**

In addition to the several monographs that have been surveyed so far, we have to mention a few more books dealing with the subjects of history of mathematics, Indian culture and world history. One of the best accounts of the ancient history of mathematics is in the monographs [37, 43]. In [37], an arduous and meticulous investigative account of the history of mathematics over a period of five thousand years, transcending the boundaries of ethnic prejudices, religious preconceptions and cultural ignorance. In [41, 45, 48], the critical

exposition of Indian history from its hoary past right up to the modern times, is given. In [44], computational error and complexity in science and engineering is discussed. One small booklet of his Telugu poems was published in 1998, in Visakhapatnam, India. He had a passion for poetry from his younger days and it never was diminished till the end. The proceedings of the several international symposia, conferences, and two world congresses (1976, 1978, 1979, 1981, 1983, 1985, 1987, 1994 and 1997) are edited by Lakshmikantham. See [8-13, 15, 28, 33]. The proceedings of 3<sup>rd</sup> and 4<sup>th</sup> world congresses are online or in digital format. In summary, to say that professor Lakshmikantham is a prolific contributor to the domain of mathematics in general and nonlinear analysis in particular will be an understatement. He has made seminal contributions to the theory and practice of differential and integral inequalities, comparison theory and Lyapunov-like energy methods in order to investigate both the qualitative and quantitative properties of nonlinear systems ranging from ordinary to set differential systems, fuzzy and fractional differential systems. His mentorship of several generations of students and colleagues has made a significant contribution to the mathematical community. We express our gratitude and heartfelt reverence to this great soul. Due to the limitation of space, we are giving the references only to monographs and books, and a few other selected references that are used in this survey [4'-8'].

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## **New trends in the theory and practices of the investigations of the solids systems with friction dynamics**

European Solid Mechanics Conference  
The mini-symposium “Dynamics of the Solids Systems with Friction”  
(July 9-13, 2012, Graz, Austria)

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In recent years there has been a surge of interest in the dynamics of systems of solids with friction under conditions of the combined kinematics. Fact is that researchers in the field of dry friction has long been known that in case of combined of the kinematics, when the rubbed solids are participated, simultaneously, in the sliding, spinning and rolling motion, the use of the classical Coulomb's law is not correct, and the friction law is undergoing significant changes.

One of the first attempts to describe the relationship of friction and spinning in the case of non-point contact of moving solids was undertaken by Contensou Assuming that the two contact surfaces are locally spherical, and the distribution of contact pressure in the contact area is determined by the Hertz' law, Contensou got numerical dependence of the dry friction force from the ratio of the slip velocity to the linear velocity of spinning. However, Contensou restricted by the friction force calculation only, without considering the friction torque.

The principle new development of the theory was given by Zhuravlev in [2]. With the aid of transition of the coordination system origin, to the instantaneous center of velocities, he obtained exact analytical expressions of the resultant vector and friction torque for circular contact spot, on the assumption that the distribution of contact pressure in the contact area subject to the law of Hertz. To use the obtained relationships in the dynamics problems, Zhuravlev built their fractional-linear but Pade. The convenience of the use of the Pade approximations making it possible to describe the effects of the combined dry friction for the entire range of angular and linear velocities resulted in the development of principally new models of friction on their basis.

Shortly after the publication of [2] in Russia and some European countries, several research groups focus their attention on studying the effects of dry friction in the combined kinematics. Main subjects of their research were the construction of mathematical models of dry friction, explaining the conditions of equilibrium of solids with dry friction and solution of various problems of classical dynamics. The narrow field of theoretical mechanics that began in the early 2000's with a few publications by Zhuravlev [2], Ivanov [3], Kireenkov [4] and Leine [5], in 2012 formed in integral scientific direction. Were published in dozens of publications on the subject. Created a need for direct exchange of new ideas and results between different research groups and to sensitize the wider scientific community.

To this aim, at the scope of 8<sup>th</sup> European Solid Mechanics Conference (ESMC8) A.V.Karapetyan and A.A.Kireenkov were firstly organized a mini-symposium on “Dynamics of the Solids Systems with Friction”. Choosing of this conference to host a mini-symposium was due to the fact that this forum, held every three years under the auspices of the European Society of Mechanical "EUROMECH" ([www.euromech.org](http://www.euromech.org)), brings together scientists and engineers from around the world conducting theoretical and applied research in all areas of the mechanics of solids. Thus, in the mini-symposium were attended specialists not only in field of the theoretical mechanics, but also in the theory of elasticity, numerical methods and various engineering applications.

Despite on the nature of the interdisciplinary mini-symposium character, his main theme was the construction of the dry friction mathematical models in the case of combined kinematics and solutions with there aid of the various dynamics problems.

Originally for oral presentations were selected 23 papers submitted by scientists from Russia, Austria, China, Serbia and Switzerland. Unfortunately, from the financial problems, in particular, because of the high registration fee is almost half of the participants from Russia and Serbia could not attend. Therefore, in the final part of the program, consisting of two sessions, only twelve (12) papers were included. The main topics of the first session were the construction of the combined dry friction models and investigation of the related effects, but the second - the solution of various problems of the dynamics of solids in systems with friction.

The right to open the mini-symposium was provided to scientists from Austria. Professor Alois Steindl, from the Vienna Technological University took lecture «Numerical Calculation of Slip-Stick Rotating Waves Caused by Coulomb Friction». His report was devoted to the study of the causes of sound vibrations that occur in the brake pads of trains due to Coulomb's friction. As a mathematical model of brake pads was considered a thin elastic cylinder. Sound vibrations in the cylinder are modeled in the form of elastic waves moving in the material of the cylinder, for the mathematical description of which required camping drawing system of two equations in the part derivations of the third order. To simplify these equations, a single-mode Galerkin method was used. Investigation of the stability of the simplified system was carried out with aid of numerical methods.

During the discussion, co-chair of the mini-symposium, it was noted that these sound waves arise and find practical application in the theory of the Hemyspherical Resonator Gyro. According to this theory, developed and Institute for Problems in Mechanics of RAS, by V.Ph.Zhuravlev the shell is modeled as thin inextensible ring, elastic vibrations of which are described by two coupled pendulum Foucault. This approach can effectively use in describing of the sound vibrations in the brake pads. Moreover, using of inextensibility condition could be quiet natural also in the description of these phenomena.

Opening of the mini-symposium by the report with strong engineering orientation with practical application in industry emphasized its interdisciplinary nature. However, since the main focus of a mini-symposium was developing of the dry friction models and investigation of the study emerging phenomena in the case of combination of kinematics, a key lecture delivered by one of the co-chair A.A.Kireenkov, was devoted to this issue. In his report «Theory of the Multi-component dry friction» were presented not only new results, but also provides an overview of a new approach to building models of dry friction under conditions of combined kinematics, developed by the author in close cooperation with his teacher, academician V.Ph.Zhuravlev.

The main distinguishing feature of this approach is that, at first, under the assumption of validity both the classical Coulomb's law in differential form for small element of the area inside the contact area and its generalized forms, there are constructed the exact coupled integral dry friction model, obtained by integrating the differentials of the principal vector and torque on the contact spot.

To this exact form, to facilitate its use in the dynamics problems, Pade expansions, based on the analytic properties of integral models, are applied. In turn, the combined dry friction models, based on Pade expansions are interpreted as phenomenological, as their coefficients can be determined from the experiment, that completely avoids the necessity to calculate the double integrals over the contact spot.

It is worth explain the used "Exact integral model" because any model can not be exact, because it is only an approximation to a real phenomenon. This notion is used in the sense

that, after the initial assumptions about the validity of Coulomb's law in classical and generalized differential form and general properties of the normal contact distributions inside of contact spot, all other computations, from a mathematical point of view, are being made exactly, without the use of approximate methods. Thus, after writing expressions for the differentials of the dry friction principal vector and torque, all subsequent transformations are exact results, reflecting the nature of the phenomenon.

We should focus on the use of Padé expansions in the theory of multi-component dry friction. Pioneered by V.Ph.Zhuravlev, to approximate the exact expressions for the friction force and torque in the problem of rolling a ball on a rough surface, with the development of the theory, they have become an indispensable instrument for modeling the dry friction effects in the case of combined kinematics.

To formalize the development theory, in the dependence on the number of kinematic parameters, determined by the force state, in [6] was introduced the concept of dimension model of dry friction, and in depending on the order of used Padé approximants - the concept of the model order. For instance, at the case circular areas of contact with a centrally symmetric distribution of contact stresses two force factors - friction force directed against the sliding velocity, and friction torque are depend on two kinematic factors: angular and linear velocity – V.Ph.Zhuravlev's two-dimensional model of the spinning and sliding friction. If the contact spots form different from the circular, additionally there is a nonzero component of the force at right angles to the direction of the instantaneous slip, and the dependence of the rotation angle of the contact spot - a three-dimensional model. Final formalization of the notion of model dimension of the multi-component dry friction was given by V.Ph.Zhuravlev in [7].

Two-dimensional friction models developed in [2, 4, 6] were constructed under the assumption that the Coulomb's law is valid in differential form for a small element of area inside the contact spot. Their extension to the case of a more real friction characteristic was given in [8], where it was shown that, in the case of combined kinematics, using of the Coulomb's law in the generalized differential form leads to a qualitatively new properties of the friction force as function of the sliding and spinning velocities, but does not change the dimensions of the model. The necessity to use instead of the classical Coulomb's law in differential form, its generalized form has been confirmed by numerous experiments performed on installation specially designed to this goal [9, 10].

The assumption that in the case of circular contact spots the normal contact stress distribution depends on the radius vector from the origin at the center of the contact, well suited for modeling the effects of the combined dry friction, but can not account for the contribution of shear stresses on the distribution of the normal pressure in the contact area. At the same time, from the works of the researchers in the field of theory [11], it is known that even a pure sliding or rolling cause the displacement of the symmetric form of the normal contact stresses distribution in the direction of rolling or sliding. In the case of combined kinematics, an additional shift in the direction perpendicular to the instantaneous velocity of sliding is arising. A distortion in the symmetry of the normal contact stress distribution in the case of circular contact spots leads to appears the friction force component directed perpendicular to the instantaneous velocity of the slip, which is identically equal to zero in the presence of symmetry.

Another important fact that affects on the construction of a friction model is a dynamic coupling of components that determine the force state of rubbed solids, appearing in case of combined kinematics due to skew in symmetric distribution chart of the normal contact stresses. Method of this coupling calculating for circular areas of contact at simplistic view of the uniformity of the normal pressure distribution inside of the contact spot was developed by

A.P.Ivanov [12]. A similar approach was used by V.Ph.Zhuravlev in studying the phenomenon of “shimmy” [7].

At the A.A.Kireenkov’s report generalization of the model of the sliding, spinning and rolling friction was presented took into account both the dynamic coupling between components that determine the force state of the rubbed solids, as well as a generalized differential form of Coulomb's law [6]. A several types of models: the exact integrated models and their analytical approximations based on Pade expansions of the first and second order were proposed. Besides, there were first constructed multi-component friction models using Zhuravlev’s approximations, based on the ratio of the linear form of the square root of quadratic form. Models of this kind can be used to solve a number of the dynamics problems instead of models based on Pade expansions as avoiding the use of non-smooth functions of the module type in the right-hand side of differential equations.

In conclusion of the lecture, the results of the first experiments on the friction model verification were presented. They have been undertaken at the Institute for Problems in Mechanics of RAS by author together with V.V.Filatov on installation, specially designed for this purpose by V.V.Filatov. These experiments clearly demonstrated the correctness of the theory of the multi-component dry friction.

It should be noted that the holding of this unique experiment was made possible by financial support from the Department of Theoretical Mechanics of MIPT and organizational support by director of IPM RAS, academician F.L.Chernousko.

Next lecture of the German scientists Matthias Graf and Mr. Georg-Peter Ostermeyer from Institute of Dynamics and Vibrations of the Technische Universitaet Braunschweig «Thermoelastic Instabilities In Layered Structures: Semi-analytical Strategy» was again devoted to the construction of the friction models, effectively working in engineering applications. The authors proposed a three-dimensional model to describe the thermal effects occurring in the brake discs. Novelty of their proposed approach was that the first attempt to link the effects of heat distribution in perpendicular directions: slip pads and in the axial direction. Previously, it was used a combination of two-dimensional and one-dimensional models, describing, respectively, the heat propagation in perpendicular directions within the contact area and in the radial direction. Using the reduction of a two-dimensional model, with subsequent numerical study, the authors constructed a two-dimensional temperature field in the brake disc.

After that, word was given to the representative of the scientific school of A.V.Karapetyan - one of the mini-symposium cochair. Maria Munitsyna from Moscow State University read report «Force and Moment of a Friction in the Case of Flat Elliptic Contact of a Body with a Supported Plane», in which it was submitted generalization of two parametric A.V.Karapetyan’s model [13].

The main feature of the A.V.Karapetyan’s model is the consideration in theory combined friction not only the form of the contact and relationship between kinematic parameters, but also an account of the elastic properties of the rubbing solids, leading to indentation of one to another. Generalization presented by Munitsyna consisted in describing the transition from circular areas contact to the elliptical. She has been constructed the integral model, for simplification of which the linear fractional Pade were used.

Utilization Pade expansions not only representatives of the Zhuravlev’s school but also with representatives of the independent Karapetyan’ school demonstrate the fact that it is a model based on these expansion become a kind of standard in the theory of combined dry friction.

The first meeting was closed by report «A Finite Element Contact Implementation for Visco-elastic Solids on Rough Surfaces» of scientists Thang Xuan Duong and Roger Andrew Sauer from Rhine-Westphalia Aachen University. In this report, it was submitted the another

approach to the construction of friction models - numerical experiment. Using the finite element method the authors have been analyzed the effect of the hysteresis properties of the material due to friction. Used by their approach also allows to investigate the influence the thermal stress effects caused by the energy dissipation in the material on its hysteresis properties. As an illustration of the developed method there were presented the results of numerical experiments showing the effects of the material parameters, such as the relaxation time, the normal load and the characteristics of the surface roughness on the built friction model for the selected material.

The second meeting of mini-symposium was opened by the report of Russian scientists dedicated to the urgent technical problem arising in aviation - the phenomenon of "shimmy". S.I.Zhavoronok from the Institute of applied mechanics of RAS with A.A.Zagordan and N.S.Bernikova from Moscow Aviation Institute delivered lectures «Main Landing Gear's Shimmy Models Based on Multi-Component Dry Friction».

The results description provided by the report authors is not possible without a brief description of the history of the shimmy phenomenon research. Generally accepted is the idea that the cause of shimmy is the deformation tire. It is assumed that the rate of wheel slip relative to the road at one point of the contact is zero (the so-called non-holonomic condition). Nevertheless, this is not the only reason. Shimmy phenomenon can be observed in everyday life, for instance, in the case of a variety of manual wheelchairs, where the reference to the tire elasticity is often inappropriate. Ignoring dry friction caused researchers to explain the shimmy was undeveloped ideas about this friction to the present, and to explain the shimmy in the previous ideas failed.

A principle new explanation of a shimmy was given by V.Ph. Zhuravlev. He has shown that the theory of multi-component friction quietly explain the phenomenon of shimmy for hard wheels and, therefore, is, at least, one of the main causes of this phenomenon in general. In particular, Zhuravlev found that if rolling friction dominates under the sliding friction then selfoscillations in the plane of rolling might be appear.

Authors of the report applied the Zhuravlev's theory to the study of phenomena high frequency oscillations of the main aircraft wheel gear aircraft on the initially landing stage. This phenomenon is different from the classical wheel shimmy, first, the unsteady nature of the rolling wheel with pronounced progressive slippage, secondly, the high velocities with rates of 1.5 - 1.8 times greater than the critical speed the nose wheel shimmy, thirdly, a higher frequency.

It was shown that unstable rolling of the main wheel landing gear is appear mostly under a slight damper compressing and small deformations pneumatics. Under these conditions, the use of the Keldysh's classical shimmy model and similar models based on non-holonomic rolling condition at zero slip is unfounded.

In the report, a new model of unstable rolling of the aircraft main wheel landing gear aircraft in the presence of slip in the contact area on the basis of the theory of multi-component dry friction Contensou-Erissmanna neglecting deformations of the wheel pneumatic was proposed. It was considered a standard unbalanced scheme of the rigidly fixed landing gear with one wheel.

Based on the results of numerical integration of the equations of motion of the wheels, the authors show you that these high-frequency oscillations shimmy landing gear with the longitudinal and lateral removal arising due to internal resonance in the system can be excited by dry friction at the end of the promotion of the wheel at the start of run. Also there were obtained of the stability region boundaries of the main landing gear of the symmetrical scheme under the influence of dry friction forces in the linearized problem formulation.

The next report «Free Nonlinear Dynamics of a Heavy Mass Particle along Rotating Rough Curvilinear Lines with Amontons-Coulomb's Type Friction and Different Rough Contact Surface Realizations», Katica R. Stevanović) Hedrih from Serbia was devoted to the another aspect of modeling of the dry friction effects – investigation of the equilibrium conditions of the classical mechanical systems.

The author carried out a comparative analysis of dynamics of the conservative mechanical system with ideal geometry and non ideal types of Amonton-Coulomb forces constraints. As an example of such a system was considered a heavy particle moving along a curved ideal line, which is rotating, in turn, about a fixed axis. The stability conditions of such systems have been investigated and constructed the corresponding phase portraits.

The rest reports included in the session program were presented by the participants from Russia.

In the report «Walking Robot Dynamics on a Rough Inclined Cylinder» of the E.F.Golubev from the Institute of the Applied Mathematics RAS and E.V.Melkumova from the Moscow State University, investigations on the effect of dry friction on the dynamics of many foots robot were presented. The robot has been simulated by the free solids having six degrees of freedom. Equations of the robot motions in space were derived, on the assumption that it can perform linear motion in three independent directions in space, and spatial rotation around a fixed point. Conditions of reactions of the robot foots that are modeled by elastic rough cylinders have been obtained. More detailed study of the problem was carried out for the two foots robot. It has been shown that in this case, the study of reactions of the robot foots can be carried out by analyzing the system of inequalities. The classification of the robot foots provisions were given in the dependence on the coefficient of friction.

Following two reports of scientists from the Moscow State University were devoted to the study of the classical problem of dynamics - free sliding with rotation of a heavy circular cylinder on a rough plane. In work of Tatyana Salnikova and Dmitry Treshchev, the plane is assumed horizontal but in work of Anna Rusinova – incline. Both works have been devoted to finding the trajectories of the cylinder and the stopping conditions. Studies were carried out using coupled integral friction models constructed in assuming a uniform distribution of normal contact stresses inside of contact spots. It is this assumption led to a lively discussion and criticisms from the mini-symposium co-chair, A.A.Kireenkov.

One of the first works, in which there was carried out a detailed study of the dynamics of a heavy circular cylinder on a rough plane under the assumption of uniform distribution of normal contact stress inside of the contact area, was made by a group of Russian scientists under under the leadership of A.Yu.Ishlinsky [14]. In this study has been shown that velocities of the sliding and spinning vanish simultaneously, and the instantaneous center of velocity is inside of the contact circle at a distance of  $0,71$  the cylinder radius.

With that, a specialists in the field of theory of elasticity are well known that distribution of normal contact stress in the case of the indentation of cylindrical punch into an elastic half-space distribution of is obeys by Galin law. However, the use of the real law of the normal contact stresses distribution in the dynamics problems was complicated by cumbersome methods of investigations. Progress was made only after the establishment of the theory of multi-component friction.

In 2002, it was published Kireenkov's paper [4], in which the dynamics of heavy disk on a rough plane in the case of combined kinematics were investigated in assumption that distribution of normal contact stresses is described by Galin law. With the aid of Zhuravlev's method of translation of the coordinate system origin to the instantaneous center of velocity, the author has obtained exact expressions for the torque and friction, expressed in terms of elementary functions, and for the ease of use of these expressions in the dynamics problems,

following to the Zhuravlev's procedure, their linear fractional Pade aproximants have been constructed. Using research method applied in Ishlinsky work [14], Kireenkov established a new qualitative fact that at the moment of stopping, the instant center of velocities is from the center of the disc to a distance exactly equal to its radius.

Kireenkov's results were repeated by N.N.Dmitriev in [15]. Furthermore, N.N.Dmitriev obtained dependence of the distance being in the moment of stopping of the instant center of velocities from the type of the normal contact stresses distribution, described by Bussenesq laws.

In connection with the above, it is surprising that many authors continue to use a non realistic view that distribution of the normal contact stress is uniform which is in poor agreement with the theory of elasticity results. Even more surprising is the active citation the results of the work [16], Hungarian scientist Farkas, published lately then Zhuravlev's and Kireenkov's works without the appropriate references.

However, it is worth noting that, as the report of Salnikova-Treshchev and the report of Rusinova contain very interesting from a mathematical point of view results.

The second session of the minisymposium concludes report of the another representative of the Moscow State University. Olga Sentimova took lecture «Dynamics of Sphere on a Horizontal Plane with Friction», devoted to the classic problem of dynamics - rolling a heavy ball on a rough plane. Using qualitative and numerical analysis, the author explains the ball trajectory divergence from a straight line in the scope of the two-parameter Karapetyan model.

Unfortunately, initially declared in program reports «On the Stability of the Motion of Railway Wheels with Dry Friction» by of the Russian scientist G.M.Rosenblatt from Moscow Road Institute and «Vibro-Impact System Based on Oscillator, with Two Heavy Mass Particles Moving along a Rough Circle» of the Serbian scientists Srdjan V.Jovi and Vladimir Raievi did not take place from the absence of participants. Serbian scientists have not arrived from the VISA problems, but Rosenblatt has not informed the mini-symposium co-chairs about circumstance his absence.

In the discussion of the mini-symposium work are active participated not only the speakers, but other conference participants. It was made numerous suggestions on continuation of conducting of special events on the mini-symposium problems with the expansion of its subjects.

Based on the colleague's wishes, Karapetyan and Kireenkov have taken decision to continue the mini-symposium work in 2013. With this aim in framework of the «International Conference on Computational Methods for Coupled Problems in Science and Engineering» [HTTP://congress.cimne.COM/2013coupled/Frontal/default.ASP](http://congress.cimne.COM/2013coupled/Frontal/default.ASP) , which will be held from June 17 to June 19, 2013 on the Ibiza island, Spain, they organized the Invited Section «Coupled Problems of the Dynamics of the Solid Systems with Friction», but in framework International Conference on Vibrations Problems (<http://www.icovp.com/>), which will be held from September 9 to September 14, 2013 at Lisbon, Portugal – mini-symposium «Oscillations in the solids systems with combined dry friction».

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**Alexey Albertovich Kireenkov**, a graduate of the Moscow Institute for Physics and Technology (MIPT), candidate of the physics and mathematics sciences, senior researcher of the Institute for Problems in Mechanics of the Russian Academy of Sciences in which he is continuously working after MIPT's graduation study and Ph.D. thesis defense. In parallel with the scientific activities, A.A.Kireenkov is actively engaged in teaching activity in leading Moscow Institutes. He takes annual lectures and conducts workshops on differential equations and the theory of functions of a complex variable. A.A.Kireenkov, the author of numerous scientific papers on the theory of combined dry friction, an active participant in international conferences, both as a author and organizer. His research interests include the theory of constructing of the dry friction mathematical models in a case of combined kinematics, investigations of the solids systems with friction dynamics, as well as research methods of non-linear vibrations of complicated mechanical systems.

**Problems of continuum mechanics**  
Scientific Seminar and Final Scientific Conference'2012  
(Kazan, 2012-2013)

**D.A.Gubaidullin**  
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The Scientific Seminar “Problems of Continuum Mechanics” works in the Institute of Mechanics and Engineering, Kazan Science Center RAS. The Chairman of the Seminar is Director of the Institute Corresponding Member of the RAS D.A.Gubaidullin. Problem reports and dissertations of researchers of the Institute and scientists from other organizations are presented and discussed at the Seminar.

The reports, representing the achievements in the sphere of non-linear mechanics of thin-walled constructions, hydroaeroelastic and wave systems; dynamics of multiphase multicomponent media in porous structures and technological installations; non-linear stability theory of control systems with changeable structure, were shown on the section of the Final scientific conference of 2012.

During the period from February 2012 to April 2013 the (Ph.D theses) in mechanics of liquid, gas and plasma and mechanics of deformable solid body, were presented at the Seminar.

The abstracts of the reports are presented below.

**The reports at the Final conference**

February 11, 2013.

**D.A. Gubaidullin, R.G. Zaripov, L.A. Tkachenko** (IME KazSC RAS).

**Forced nonlinear oscillations of aerosol in an open tube in the shock-free mode.**

The coagulation and deposition of small-disperse aerosol under excitation by the flat piston in an open tube in the shock-free mode near to the first natural frequency are experimentally studied. Dependences of oscillations of pressure at the piston and number concentration of drops of oscillating aerosol from time are received. The influence of frequency and amplitude of oscillations on the time of aerosol clearing is studied. It is shown, that even at small displacement amplitudes of the piston the process of the aerosol clearing, consisting in coagulation, deposition on walls of a tube and partial emission of drops of an aerosol from the open end of a tube, occurs 6-12 times more effectively, than natural deposition.

**D.A.Gubaidullin, A.A. Nikiforov, E.A. Teregulova** (IME KazSC RAS).

**The propagation of acoustic disturbances in multifractional gas suspensions.**

Propagation of acoustic waves in the mixture of gas with fractions of particles of different materials and sizes is investigated. The mathematical model is presented, the dispersion relation and a wave equation are received, and dispersion curves are calculated. The influence of many-fractionousness of a composition of a disperse phase, parameters of a disperse phase for a many-fractional gas mixture with particles of ice, sand and soot on a dissipation and a dispersion of acoustical waves is analyzed.

**D.A. Gubajdullin, A.A. Nikiforov, R.N. Gafiyatov (IME KazSC RAS).**

**Acoustic waves of different geometry in two-fraction bubble liquids.**

The propagation of acoustic waves of different geometry in two-fraction mixtures of liquid with vapor–gas bubbles of different sizes and compositions with phase transformations in each fraction has been studied. A system of the differential equations of the motion of the mixture is presented, and the dispersion relation is deduced. The evolution of the weak pulsed perturbations of the pressure in this mixture was calculated numerically. An effect of phase transformations in each fraction of disperse phase on evolution impulse of pressure is shown.

**D.A. Gubaidullin, Yu.V. Fedorov (IME KazSC RAS).**

**Acoustic waves of different geometry in polydisperse bubbly liquids.**

The propagation of acoustic waves in two-fractional mixtures of a liquid with polydisperse gas bubbles of different structure is investigated. The system of the differential equations of the perturbed motion of a mixture is presented, the dispersion relation is obtained. The comparison of the developed theory with known experimental data is presented.

**D.A. Gubaidullin, P.P. Ossipov (IME KazSC RAS).**

**About the universal diagrams of the intensity of wave force in the problems of the drift of entrainments.**

Various models and mechanisms of drift inclusion in the standing wave are studied. The impact of inclusion density and entrainment coefficient on the direction and intensity of the wave force are investigated analytically and numerically. The chart of intensity and direction of the wave force is suggested.

**A.L. Tukmakov (IMM KazSc RAS).**

**Model of movement and sedimentation of the loaded gas-suspension in electric field.**

In work the model of dynamics of the loaded gas-suspension moving in electric field under the influence of power of interphase friction and the electrostatic force is constructed. The model includes the equations of movement of the bearing and disperse phase, written down taking into account an interphase exchange of an impulse and energy. The system is solved by the predictor-corrector method with the spatial directions splitting. The method is applied to investigate the fields of speed and gas-suspension density in interelectrode space and near a surface on which there is a dusting.

**D.A. Gubaidullin, D.A. Tukmakov (IME KazSc RAS).**

**The influence of the solid fraction characteristics on the process of the break-up in the gas mixture.**

In this work the influence of the various characteristics of gas suspension on its dynamics is considered. The process of the break-up in the gas mixture if the hard phase has different density, volume content and size of dust particle was researched. For the modeling of the gas mixture motion the system of equations of the two speed, two temperature gas mixture without phase changes was applied. The carrying agent was the viscose, heat-conducting gas. The authors took into account the intensity of interphase friction and interphase heat-conductivity. The system of the equations was solved by the McCormack method of the second order of accuracy. With the help of the numerical modeling it was revealed that the process of the break-up in the gas mixture differs from the same process of the break-up in the free gas.

**V.G. Tonkonog (KNRTU), D.A. Tukmakov (IME KazSc RAS).**

**Separation of the gas mixed in the acoustic resonator of open type.**

In this work fluctuations and separations of the gas mixture in the traffic of acoustic resonator are investigated.

The acoustic resonator was the plane canal in which the direction of gas fluctuations is perpendicular to the directions of the traffic of heterogeneous environment. The carrying agent was the viscous, compressible, heat-conducting gas. The equations of dynamics were written in the two dimension case taking into account the interphase frictions and heat exchange. The numerical modeling of various processes in the acoustic resonator was made the conditions of the drift of the hard phase in the dispersed flow under the influence of the nonlinear acoustic fields were defined. The results of numerical calculations of the gas mixture dynamics and the distribution of the density of dispersion phase in dependence on the carrier phase speed at different time moments were given. The results confirm the possible creation of technology of acoustic separation of the gas mixed.

**A.A. Aganin (IME KazSC RAS).**

**Cumulation under collapse of cavitation bubbles in liquid.**

Main results obtained in the course of investigating the collapse of cavitation bubbles in liquid in the laboratory of Numerical dynamics of continuous medium IME KazSC RAS in 2012 are presented.

**A.A. Aganin, N.A. Khismatullina (IME KazSc RAS).**

**Elastic-plastic deformations in a body under the impact of liquid jet to its surface.**

A numerical technique of investigation of elastic-plastic deformations in a body under application to its surface a load simulating pulse action of a high-speed cumulative liquid jet arising during collapse of a cavitation bubble attached to the body is developed. Position, size and configuration of yielding zones arising in the body as well as their dependence on plasticity of material and radially non-uniformity of the load are studied.

**T.S. Guseva (IME KazSC RAS).**

**Numerical simulation of fluid dynamics without explicit separation of contact interfaces.**

A numerical algorithm for computer simulation of problems of contact interaction of gas-liquid type with large deformations of the contact boundary and with shock waves present in the media has been developed. Main ideas underlying the algorithm are based on the CIP-CUP approach using Euler meshes without explicit separation of contact interfaces and governing equations of compressible fluid flow in terms of non-conservative variables. Comparison of results of computation of test-problems with analytical solutions and results computed by other authors is given. The capability of the technique realized is illustrated by its application to problems of impact of axially-symmetric high-speed liquid jet on a rigid wall and on a layer of liquid on the wall.

**A.I. Davletshin (IME KazSc RAS).**

**Determination of the velocity potential of a liquid with weakly non-spherical inclusions located in line.**

A generalization of the method of reflections intended for calculating the potential of liquid velocity with two spherical inclusions in case of the presence of an arbitrary number of weakly non-spherical inclusions the centers of which are located in line has been made. The recurrent relations obtained have been tested. Comparison of consumption of computer time

to evaluating the velocity potential by the method of reflections and the method of decomposition in spherical functions has been performed.

**L.A. Kosolapova, V.G. Malakhov** (IME KazSc RAS).

**Collapse of a cavitation bubble in a fluid near a rigid wall.**

Dynamics of an axisymmetric cavitation bubble during its collapse in the liquid near a plane rigid wall till the moment of its destruction beginning (loss of simple connectedness) is considered. Deformation and displacement of the surface of the cavity and the change of the liquid velocity are calculated by Euler scheme using the boundary element method. Dependence of the features of the collapse on the distance of the cavity from the wall and the initial distortion of her spherical shape is investigated. In the case of initial ellipsoidal deviations of the cavity shape from the spherical one, the range of values of the ratio of semi-axes for which a cumulative jet directed perpendicularly to the wall is formed on the cavity surface has been determined.

**D.Yu. Toporkov** (IME KazSc RAS).

**Features of a cavitation bubble compression in benzol.**

Strong compression of a cavitation bubble in benzol has been investigated. To describe the motion of the liquid and vapor, a mathematical model with decomposing into spherical component and a small non-spherical perturbation is used. The spherical component is described by the gas dynamics equations taking into account the transient heat conduction, non-equilibrium evaporation-condensation at the interface and the realistic equations of state. A shock wave has been shown to form in the bubble. It was found that during compression of the bubble the growth of its nonsphericity does not exceed 180 times if the initial radius of the bubble is more than 100  $\mu\text{m}$ .

**M.S. Ganeeva, V.E. Moiseeva** (IME KazSc RAS).

**About the effect of a type of loading on the process of spherical shells nonlinear straining.**

In this work the process of geometrically and material nonlinear straining of spherical shells under the action of axisymmetric and nonaxisymmetric loading with determination of limit loads of instability is studied. The results of numerical calculations depending on geometrical characteristics of shells, material properties and a type of loading were obtained.

February 12, 2013.

**A.V. Elesin, A.Sh. Kadyirova** (IME KazSc RAS).

**Identification of the absolute permeability on the measurements of debit on the well under the conditions of two-phase filtration.**

Algorithms for the solution of the identification task of the absolute permeability on the measurements of debit on the well working at a given pressure are proposed. These algorithms are based on the standard and two-step Levenberg-Marquardt minimization methods.

**A.V. Tsepaev** (IME KazSc RAS).

**Solution of three-phase flow equation by decomposition methods.**

Numerical algorithms to solve the three-dimensional three-phase flow equation based on the methods of decomposition have been developed. These algorithms show their efficiency on

heterogeneous computing systems. The numerical approximation is based on the control volumes methods.

**G.A. Nikiforov** (IME KazSc RAS).

**About two-phase vortex flows.**

The problem of isothermal two-phase filtration of two immiscible incompressible fluids in the variables "velocity-saturation" is solved. The vortex flow in porous solids taking into account the capillary and gravitational forces is studied.

**A.I. Nikiforov, T.R. Zakirov** (IME KazSc RAS).

**The influence of filtering modes on efficiency of acid treatment on the oil reservoir.**

The problem of the influence of the injection rate on the effectiveness of the reagent acid exposure to the oil reservoir. The model as a bundle of cylindrical capillaries of various radii is used to describe the changes in the porosity and properties of the porous medium. The system of equations is solved by volume method on a uniform grid.

**P.E. Morozov** (IME KazSc RAS).

**Filtration pressure waves in the vicinity of the wellbore with the crack of hydraulic fracturing.**

New analytical solutions are presented for analyzing amplitude- and phase-frequency characteristics of fractured wells in homogeneous or double-porosity reservoirs. The influence of wellbore storage effect, fracture storage and conductivity on the pressure modulus and phase shift is investigated. In case of high dimensionless frequencies a set of asymptotic solution is derived. These solutions can be used to solve the inverse problems for obtaining the formation and fracture properties.

**V.R. Gadilshina, M.H. Khairullin** (IME KazSc RAS).

**Thermodynamic processes in the vicinity of the vertical wellbore before and after hydraulic fracturing.**

In this paper, a mathematical model to describe nonisothermal filtration in the vicinity of the vertical well before and after hydraulic fracturing is constructed. The influence of thermal and filtration parameters on the temperature variations in the bottom hole after bringing well into production is analyzed. A computational algorithm for processing the results of measurements in the borehole is proposed. The proposed algorithm is used for interpretation of the temperature and pressure curves recorded in the well no. 2030.

**B.A. Snigerev** (IME KazSc RAS).

**Two layer immiscible of non-newtonian fluid flow with free surface.**

Numerical simulation of the double-layer flowing of nonlinear viscous fluid from flat channel was performed. The fluid motion is described by conservation equations of mass and momentum, supplemented by the rheological equation of state of nonlinear viscous fluid on the Carreau model. The technique of numerical problem solving based on the finite element method is described. The Results of the shape of free surface, velocities of fluid, pressures and stresses in a two-layer flow depending on rheological properties of liquid and flow regimes are presented.

**V.L. Fedyaev, I.V. Morenko** (IME KazSc RAS).

**Non-isothermal fluid flow in the annular channel with a twist at the laminar regime.**

A laminar flow of a viscous incompressible fluid in the annular channels with spiral-shaped wire inserts was considered. The fact that the Heat transfer from the outer wall of the channel is greater than from the inner one was established. With increasing Reynolds number The heat transfer of concave wall is growing faster than of convex one. The dependence of the friction coefficient of the channel of the average surface temperature of the walls of the Nusselt number on the Reynolds number was founded.

**N.Z. Serazetdinov, V.L. Fedyaev** (IME KazSC RAS).

**Fluid flow in cavities of main pipeline filter.**

Numerical calculations of hydrodynamics of fluid in cavities of filters were carried. Various configurations of filters have been considered. The dependence of the flow pattern on the configuration of the filter is analyzed.

**E.M. Vlasov** (IME KazSC RAS).

**Calculation of refrigerating ability and increase of efficiency of evaporative cooling towers.**

The model of evaporative cooling of flowing down films of water on vertical plates of irrigator of cooling tower, drops falling down between them, blown by rising gust of vapour-aerosol stream is formulated. Analytical dependences for calculation of temperature difference gradient of water are presented. For the operational efficiency of cooling towers the approximate dependences for the definition of auxiliary parameters of generic calculation of cooling towers, ratios of temperature gradient of water and heat efficiency are estimated. High efficiency modular cooling towers, in which gradual cooling of circulating water is implemented, are developed based on received data.

**N.K.Galimov, S. N. Yakupov** (IME KazSC RAS).

**The modified elastic potential for rubber membranes.**

In practice for calculation on durability of rubber products which experience big deformations it is necessary to know their elastic potential on the basis of which and according to Green formulas it is possible to calculate tension. About ten forms of representation of elastic potentials are known. Having used Mooney's and Chernih's potentials and experimental data of tests of the rubber spherical membranes loaded with uniform pressure, the modified options of elastic potentials for which constants are defined are offered. Some results are given.

**N.M.Yakupov, H.G.Kiyamov** (IME KazSC RAS).

**Research of a fragment of a spherical cover with defect.**

In use there can be various defects on the surfaces of covers which are usually a cause of destruction of designs. According to the shell scheme of sampling it is impossible to determine concentration of tension of part-through defects through thickness. In the work, on the basis of developed in spherical system of coordinates of spline variant of FEM in three-dimensional statement the spherical covers with openings in poles which can be intact ones as well as ones with a meridional crack on the half of thickness are considered. Pictures of concentration of tension are given in defective areas.

**S.N. Yakupov** (IME KazSC RAS).

**To research of adhesion of a film to a substrate.**

Now various coverings are created and technologies of formation of a covering in system "a substrate - a covering" develop. Known ways and research approaches, and also their shortcomings are noted. In work the development of idea of the patent on the invention №2421707 which allows to investigate in two-dimensional statement adhesive properties of films on various substrates is described. Thus, besides diameter of peeling (according to the patent №2421707), height of lifting of a dome, and also geometrical and mechanical characteristics of a material of a film are considered. The Results of research are given.

**A.R.Nurgaliyev** (IME KazSC RAS).

**To research of work of designs in the corrosion environment.**

Designs, perceiving loadings, are exposed to the accelerated corrosion wear. The difficult form and structure of a near-surface layer owing to corrosion and existence of scratches and cracks causes change of properties of a material on some depth; there are difficulties at measurement of thickness of the element depending on the base of a micrometer. In this regard it is necessary to define integrated hardness characteristics of studied elements. For the site of the designs which has undergone corrosion, the given module of elasticity is defined and the TDC is analysed. The account of properties change leads to redistribution of efforts in designs.

**N.M. Yakupov, A.A. Abdyushev** (IME KazSC RAS).

**Research of the TDC of the panel with elliptic cut.**

One of the reasons of rupture of designs is the presence of concentrators of pressure. Among concentrators it is possible to note the presence of various cuts which causes the redistribution of pressure. Panels with elliptic cut with various parametres of the relation of semiaxes, and also panels with the active and passive overlays, blocking cut are numerically investigated. For the TDC analysis MFE with bilinear approximation of movings is used. By installation of overlays, especially active, in the field of defect, it is possible to influence on distribution and size of extreme pressure in the panel.

**R.R. Giniyatullin** (IME KazSC RAS).

**Statistical processing of results of research of membranes with corrosion defects.**

At an experimental research there is a question of an estimation of an error of measured sizes as there are no exact devices and methods of measurements. Using the algorithm of statistical processing of the experiment results the average values of a deflection, value of root-mean-square deviations, a confidential interval are defined for a series of gaugings. In technical calculations the confidential probability  $P = 0.95$  is accepted. The comparison of deviations shows, that all data is kept within the resulted critical intervals and it is possible to use these data for the further calculations.

**R.G. Nurullin** (IME KazSC RAS).

**The analysis of safety of designs and constructions.**

The approach of an estimation of safety of technical systems is developed, the recommendations for unification of concept of factor of safety as the relation of the indicators with the same name bringing the contribution to the maintenance of security are offered: in numerator - current value, and in a denominator - standard value. The general factor of safety is defined as an average value of factors of safety. The factor of safety of

communications between objects in a consecutive chain is calculated as product of factors of safety of communications between objects. The approach is illustrated on an example of water-cooling tower.

**N. M. Yakupov, L.U.Sultanov** (IME KazSC RAS).

**The analysis of concentration of tension in a spherical cover with a crack.**

It is known that one of the most essential causes of destruction of designs is existence of concentrates. The most dangerous concentrators are cracks. On the basis of two-dimensional approach in principle it is impossible to estimate level of concentration of tension through thickness. On the basis of three-dimensional elements with square approximation (ANSYS) researches of concentration of tension in a spherical cover with a crack length to the equal thickness of a cover are executed. It is shown that with change of depth of a crack there is a transition of the maximum tension from area under a crack to the radial edges of a crack.

**A.I. Malikov.** (IME KazSC RAS).

**Stability and stabilization of hybrid systems with switchings**

The stability conditions and the way of stabilization at dependent on a state space switchings are presented. For linear systems with switchings with big enough activity interval in each mode, the observers are offered, which allow to build the guaranteed estimations of a state space vector as to known, and unknown switching functions.

**Reports at the Seminar**

February 14, 2012

**A.R. Mangusheva** (Kazan State University of Architecture and Engineering).

**The estimate of short and long durated strength of film-tissue composition material.**

On materials of the thesis submitted for Candidate of Sciences Degree, speciality 01.02.04 – Mechanics of deformable solid. Scientific chief Prof. R.A. Kayumov. Reviewer PhD N.K. Galimov.

March 13, 2013

**M. N. Zapivakhina** (Bashkir State University, Birsk Branch).

**Fluxions in porous medium during the injection of water or gas.**

On materials of thesis submitted for Candidate of Sciences Degree, speciality 01.02.05 – mechanics of liquid, gas and plasma. Scientific chief DrSc V.Sh. Shagapov. Reviewer DrSc A.I. Nikiforov. The filtration during the injection of water or gas into a porous medium is studied in self-similar formulation. The effect of the parameters of the porous medium and the pressure and temperature of an injected substance on the length of the region of ice or gas hydrate formation (decomposition) in a porous stratum is analyzed.

Potential participants are kindly invited for the presentation of their results at the Seminar. Contact address – [gubajdullin@mail.knc.ru](mailto:gubajdullin@mail.knc.ru)

**Damir A.Gubaidullin**, the Director of IME KazSC RAS; the Deputy Chairman for research of Kazan Science Center RAS; the Corresponding Member of the RAS; the specialist in the area of mechanics and thermo-physics of multiphase media.  
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*Congratulations!*



*Gennadiy Lukich  
Degtyarev  
(on the occasion of the 75<sup>th</sup> birthday)*

**Gennadiy Lukich Degtyarev**, Distinguished Professor of KAI, Head of Automatics and Control Department of KNRTU-KAI, Russian Government Prize Laureate in the field of science and engineering, Honored Scientist of the Russian Federation, member of the Academy of Sciences of Tatarstan Republic, turned 75 on 27 January 2013.

G.L.Degtyarev is a well-known scientist specializing in optimal control, mathematical simulation, development of spacecraft control algorithms allowing for elasticity, and development of theory of adaptive optical systems.

Endowed with a talent for organization and being a patriot of Russian basic engineering science, he actively promotes careful attitude towards traditions and augmentation of profound traditions laid by the founder of Kazan Aviation Institute N.G.Chetaev, successor of A.M.Lyapunov – an outstanding developer of the theory of motion stability.

G.L.Degtyarev chaired KAI in 1987 in the capacity of a Rector; since 2007 he has been the University President. He provided intensive development of Kazan Aviation Institute (at present – A.N.Tupolev Kazan National Research Technical University). Under Degtyarev's personal supervision and with his active participation the ideas and concepts of development of KAI as a university have been implemented keeping the spirit and letter of the top-ranking National educational engineering schools, which is of primary importance for a higher school specializing in aviation.

Being an idea hamster with a foresight, scientific and engineering flair, understanding of prospects and important trends of development of Russian science and engineering education on the whole, Gennadiy Lukich has contributed much to the development of new educational technologies with effective cooperation with industrial institutions and plants of the main industrial complexes.

G.L.Degtyarev is an Editor-in-Chief of scientific and engineering journal “Vestnik KGTU im.A.N.Tupoleva” (“KSTU Bulletin”); Honorary Editor of the International editorial committee of the Russian-American scientific bilingual journal “Actual problems of aviation and aerospace systems”; Editor-in-Chief of the international scientific bilingual journal “Problems of nonlinear analysis in engineering systems”.

Best wishes to our dear Gennadiy Lukich!

Editorial Committee wishes every happiness, health, success and vigorous activity!

“Stability in all things big and small according to A.M.Lyapunov under all permanent perturbations”!

From all the members of Editorial Committee  
of the International scientific edition  
January 2013

# **INTERNATIONAL FEDERATION OF NONLINEAR ANALYSTS**

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## **INTERNATIONAL RUSSIAN-AMERICAN SCIENTIFIC JOURNAL**

### **Actual problems of aviation and aerospace systems processes, models, experiment (APAAS)**

International Russian-American Scientific Journal “Actual problems of aviation and aerospace systems” (ISSN 1727-6853) is founded in 1995 on the initiative of Scientists-Professors from Kazan Chetayev School of Mechanics and Stability, representatives of Kazan Aviation Institute – Kazan State Technical University of A.N.Tupolev’s name, (RUSSIA) and Scientists-Professors from Embry-Riddle Aeronautical University (USA).

The Journal is edited on base of Russian Centre of International Federation of Nonlinear Analysts (IFNA), under the aegis of two Academies (Academy of Aviation and Aeronautics Sciences and Russian Academy of Astronautics of K.E.Tsiolkovsky name) and Russian Federation of Astronautics, in cooperation with MAI (Moscow Aviation Institute – State Technical University), Bauman MSTU (Bauman Moscow State Technical University), ICS of RAS; with support of Education-Science Ministry of RF, of Federal Space Agency, of TsNIIMash, of TsNII “Electropribor”.

“APAAS” is International scientific periodic Journal on a broad spectrum of multidisciplinary problems in Aviation and Astronautics, including researches in areas of dynamics and flight control; theory, design and technology of aircrafts, engines; sciences on materials; information and computing systems, experimental investigations; economic and humanity problems of operation; remote sensing of Earth, information satellite technology; the problems of Higher Engineering Education; the problems of nanotechnology for Avia-, Aerospace systems, ...

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## International Scientific Edition

IFNA-ANS International Scientific Edition (ISE) is founded by Kazan Chetaev School of stability and mechanics in 1994. This interuniversity non-ordinary initiative raised by the intelligence of swiftly developing World brilliantly implements the objectives and goals laid in its foundations which had been formed by the following provisions:

- "True theory cannot be linear" (A.Einstein);
- "Unity in Diversity" (V.Lakshmikantham);
- "If to be, it is necessary to be the First" (V.P.Chkalov);
- "Newtonian Mechanics is an unequalled achievement of physics (natural philosophy), the whole history of human civilization. IT IS EVERLASTING. Its powerful tree is sprouting more and more branches. Among them there are the branches that have grown from scions grafted on this tree and cultivated in other natural sciences" (G.G.Chyorny);
- "Mathematics is an effective "transport" which is able to provide significant breakthrough in understanding of the essence of Environment, with deep penetration of its approaches into all the spheres including the unconventional ones".

The period of effective and successful activities resulted in establishment of ISE as a *bilingual* interdisciplinary Scientific Edition representing researches of nonlinear problems in all the diversity of basic and applied sciences. Structurally the Journal is organized as periodic Edition in two series (Journals), with preparing invited articles (as problematic character surveys) and also special topical issues on advanced scientific directions including natural sciences and the humanities (mathematics, mechanics, physics, chemistry, engineering sciences, including aviation and aerospace technologies; biological, medical, social and political sciences; ecology, cosmology, economics; nanoscience and nanotechnology, stability and sustainable development in economical, social and political systems; problems of risk and information protection, operational research, problems of higher engineering education, ...).

**Problems of nonlinear analysis in engineering systems** (ISSN 1727-687X) [http://www.kcn.ru/tat\\_en/science/ans/journals/ansj.html](http://www.kcn.ru/tat_en/science/ans/journals/ansj.html)

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- A.G.Butkovsky. *Some Principal Features of "Unified Geometric Theory of Control".*
- P.J.Werbos. *Brain-Like Intelligent Control: From Neural Nets to True Brain-Like Intelligence*
- S.Santoli. *Information-driven nonlinear nanoengine hierarchies for biomimetic evolware.*
- P.J.Werbos. *Order from chaos: a reconsideration of fundamental principles.*
- P.Marmet. *The overlooked phenomena in the Michelson-Morley experiment.*
- A.N.Panchenkov. *The entropy model of hydrodynamics.*
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Among the topical issues there are the *special issues* on the following topics:

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- Douglas Davidson. *Boeing in Russia.*
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- J. von Puttkamer. *From Huntsville to Baikonur: A Trail Blazed by S.P.Korolev.*
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- No.3(25), 2007 – *Special issue.* To the 50<sup>th</sup> Anniversary of the beginning of Space Era.
- No.1(26), 2008 – *Special issue.* To the 50<sup>th</sup> Anniversary of the beginning of Space Era and the Military Space Forces Day.
- No.1(32), 2011 – *Special issue.* To the 50<sup>th</sup> Anniversary of the first flight of a Man in Space (Space flight of Yu.A.Gagarin).

An outstanding ability to foresee, fundamental nature, responsibility, fine qualification in the whole diversity of the problems of science, education and applications are the main constituents providing functional success of this ISE which contributes much in development of science on the whole, promotes interdisciplinary community of scientists and researchers who work in different spheres.

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INTERNATIONAL SCIENTIFIC JOURNAL

## **Problems of Nonlinear Analysis in Engineering Systems (PNAES)**

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