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THEOREM ON UNIQUENESS OF SOLUTION OF A PROBLEM OF THE THEORY OF JOINT MOTION OF RIVER-BED WATER AND GROUNDWATER

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In this article we investigate the uniqueness of a generalized solution of the problem of the form

$$\frac{\partial \varphi_1(u)}{\partial t} - \sum_{i=1}^{2} \frac{\partial}{\partial x_i}(a_i(x,u)k_i(x,\nabla u)) = f_1, \quad x \in \Omega_\Pi = \Omega \setminus \Pi, \tag{1}$$

$$\frac{\partial \varphi_2(u)}{\partial t} - \frac{\partial}{\partial s}\Big(a_\Pi(x,u)k_\Pi\Big(\frac{\partial u}{\partial s}\Big)\Big) + \Big[\sum_{i=1}^{2} a_i(x,u)k_i(x,\nabla u)\cos(n,x_i)\Big]_\Pi = f_2, \quad [u]_\Pi = 0, \quad x \in \Pi, \tag{2}$$

$$u(x,0) = u_0(x), \quad u|_\Gamma = g(x). \tag{3}$$

Here Ω is a bounded domain of the space R^2, Π a cut inside Ω, which divides it into two connected domains, Γ the boundary of Ω, $[\cdot]_\Pi$ the jump of the function under transition over the cut Π, n the normal to Π, $\frac{\partial}{\partial s}$ the directional derivative along the direction Π.

Equations (1), (2) arise in modeling the process of filtration of the groundwater with regard for water level in an open river-bed (see, e.g., [1]). In this case Ω is a domain where the filtration of groundwater occurs, Π corresponds to the river-bed (channel), u determines the height of the free surface of liquid with respect to the zero impervious basement.

For problem (1)–(3) in [2] the theorem on the existence of a generalized solution was proved. As for the uniqueness, the question remained open even for a more particular case, where (see [1])

$$\varphi(\xi) = \xi \ \ \forall \xi \in R^1, \quad k_i(x,\xi) = \xi_i \ \ \forall \xi \in R^2.$$

The situation changed with publication of [3], where a new method for the proof of uniqueness for degenerating equations was suggested. Using the idea of the mentioned paper here we establish the uniqueness of the generalized solution for the case where the boundary conditions are stationary.

1. Statement of problem

We define a generalized solution of problem (1)–(3), following [2]. In addition, we shall use the following notation. Let V, $V(0,T)$, $W(0,T)$ be Banach spaces of functions, resulting by taking the closure of $C^\infty(\Omega)$ and $C^\infty(0,T;C^\infty(\Omega))$ in the following norms:

$$\begin{aligned}
\|u\|_V &= \|u\|_{W^1_{p_1}(\Omega)} + \|u\|_{W^1_{p_2}(\Pi)},\\
\|u\|_{V(0,T)} &= \|u\|_{L_{p_1}(0,T;W^1_{p_1}(\Omega))} + \|u\|_{L_{p_2}(0,T;W^1_{p_2}(\Pi))},\\
\|u\|_{W(0,T)} &= \|u\|_{V(0,T)} + \|u\|_{L_\infty(0,T;L_{\alpha_1}(\Omega))} + \|u\|_{L_\infty(0,T;L_{\alpha_2}(\Pi))}.
\end{aligned}$$

Respectively, $\overset{\circ}{V}$ ($\overset{\circ}{V}(0,T)$, $\overset{\circ}{W}(0,T)$) is the closure of functions finite in Ω (Q_T) in the corresponding norm. In the literature such spaces are termed the strengthened Sobolev spaces. To the study of

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