ISSN 1066-369X, Russian Mathematics (Iz. VUZ), 2010, Vol. 54, No. 10, pp. 27–37. © Allerton Press, Inc., 2010.
Original Russian Text © S.N. Il'in, 2010, published in Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika, 2010, No. 10, pp. 31–43.

Direct Sums of Injective Semimodules and Direct Products of Projective Semimodules Over Semirings

S. N. Il'in[1*]

[1]*Kazan State University, ul. Kremlyovskaya 18, Kazan, 420008 Russia*

Received December 30, 2008

Abstract—We prove that if the direct sum of a family of semimodules over a semiring S is an injective semimodule or if the direct product of a family of semimodules over S is a projective semimodule, then the cardinality of the subfamily consisting of all semimodules which are not modules is strictly less than the cardinality of S. As a consequence, we obtain semiring analogs of well-known characterizations of classical semisimple, quasi-Frobenius, and one-sided Noetherian rings.

DOI: 10.3103/S1066369X10100038

Key words and phrases: *semiring, injective semimodule, projective semimodule.*

According to [1], by a *semiring* we mean an algebraic system $(S, +, \cdot, 0)$ such that 1) $(S, +, 0)$ is a commutative monoid; 2) $(S, \cdot)$ is a semigroup; 3) $(x + y)z = xz + yz$, $x(y + z) = xy + xz$ for all $x, y, z \in S$; 4) $0x = x0 = 0$ for all $x \in S$. In the case when $(S, \cdot, 1)$ is a monoid with identity, S is said to be a *semiring with identity*. In what follows all semirings in question are assumed to be semirings with identity, not excepting the case when $1 = 0$ and, therefore, $S = \{0\}$. Obviously, each associative ring is a semiring.

Let S be a semiring. A commutative monoid $(M, +, 0_M)$ is called a *right S-semimodule*, if, for any $m \in M$ and $s \in S$, a product $ms \in M$ is defined such that, for all $m, m' \in M$, $s, s' \in S$, the following identity hold: 1) $m(ss') = (ms)s'$; 2) $(m + m')s = ms + m's$; 3) $m(s + s') = ms + ms'$; 4) $m1 = m$; 5) $0_M\, s = 0_M = m0$. In what follows all semimodules in question are assumed to be right semimodules. A semimodule M is said to be a module if each element of M has an opposite element. One caneasily see that if S is a ring, then each S-semimodule is a module.

Let M and M' be S-semimodules. A mapping $\varphi : M \to M'$ is called an *S-homomorphism* if $\varphi(m + m') = \varphi(m) + \varphi(m')$ and $\varphi(ms) = \varphi(m)s$ for all $m, m' \in M$, $s \in S$. By monomorphisms and epimorphisms of S-semimodules we mean injective and surjective S-homomorphisms, respectively. A semimodule M is called *projective* if, for any S-semimodules A and B, any S-epimorphism $\alpha : A \to B$, and any S-homomorphism $\varphi : M \to B$, there exists an S-homomorphism $\psi : M \to A$ such that $\varphi = \alpha \circ \psi$. A semimodule M is called *injective* if, for any S-semimodules A and B, any S-monomorphism $\alpha : A \to B$, and any S-homomorphism $\varphi : A \to M$, there exists an S-homomorphism $\overline{\varphi} : B \to M$ such that $\varphi = \overline{\varphi} \circ \alpha$.

In this paper, we study closedness of the class of injective S-semimodules and of the class of projective S-semimodules with respect to direct sums and direct products. As usual, for an arbitrary index set I, the direct sum of I copies of a semimodule M is denoted by $M^{(I)}$, and the direct product of I copies of M is denoted by M^I.

It is not difficult to check that the class of projective S-semimodules is closed with respect to any direct sums and that the class of injective S-semimodules is closed with respect to any direct products ([1], propositions 17.19 and 17.23). At the same time, an infinite direct sum of injective S-semimodules, in the general case, may not be an injective semimodule, and an infinite direct product of projective S-semimodules may not be a projective semimodule, even if S is a ring ([2], theorem 6.5.1; [3], corollary 20.22). Nonetheless, for example, any direct sum of injective modules over a right Noetherian ring is an injective module, and any direct product of projective modules over a quasi-Frobenius ring is a

*E-mail: Sergey.Ilyin@ksu.ru.