ISSN 1066-369X, Russian Mathematics (Iz. VUZ), 2010, Vol. 54, No. 3, pp. 7–11. © Allerton Press, Inc., 2010.
Original Russian Text © Dinh Trung Hoa and O.E. Tikhonov, 2010, published in Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika, 2010, No. 3, pp. 9–14.

To the Theory of Operator Monotone and Operator Convex Functions

Dinh Trung Hoa* and **O. E. Tikhonov****

Research Institute of Mathematics and Mechanics, Kazan State University, ul. Professora Nuzhina 1/37, Kazan, 420008 Russia

Received June 23, 2008

Abstract—We prove that a real function is operator monotone (operator convex) if the corresponding monotonicity (convexity) inequalities are valid for some normal state on the algebra of all bounded operators in an infinite-dimensional Hilbert space. We describe the class of convex operator functions with respect to a given von Neumann algebra in dependence of types of direct summands in this algebra. We prove that if a function from $\mathbb{R}^+$ into $\mathbb{R}^+$ is monotone with respect to a von Neumann algebra, then it is also operator monotone in the sense of the natural order on the set of positive self-adjoint operators affiliated with this algebra.

DOI: 10.3103/S1066369X10030023

Key words and phrases: *operator monotone function, operator convex function, von Neumann algebra, C^*-algebra.*

Let $\mathcal{A}$ be a C^*-algebra, denote by $\mathcal{A}^{\mathrm{h}}$ its Hermitian part. The cone of positive elements defines an ordering relationship on $\mathcal{A}^{\mathrm{h}}$. The functional calculus for elements of $\mathcal{A}^{\mathrm{h}}$ allows us to associate a real continuous function with a mapping from some subset of $\mathcal{A}^{\mathrm{h}}$ in $\mathcal{A}^{\mathrm{h}}$. The goal of this paper is to formulate some assertions connected with the monotony and convexity of such mappings.

In what follows the symbol Ω stands for an arbitrary interval in $\mathbb{R}$, i.e., a convex subset of $\mathbb{R}$. We denote the spectrum of an operator A or an element of a C^*-algebra by $\sigma(A)$; we do the C^*-algebra of all bounded operators in a Hilbert space $\mathcal{H}$ by $\mathcal{B}(\mathcal{H})$.

Definition. Let $\mathcal{A}$ be a unital C^*-algebra. A continuous function $f : \Omega \to \mathbb{R}$ is called $\mathcal{A}$-*monotone* [1], if any $A, B \in \mathcal{A}^{\mathrm{h}}$ such that $\sigma(A), \sigma(B) \subset \Omega$ and $A \le B$, satisfy the inequality $f(A) \le f(B)$. A continuous function $f : \Omega \to \mathbb{R}$ is called $\mathcal{A}$-*convex*, if any $A, B \in \mathcal{A}^{\mathrm{h}}$ such that $\sigma(A), \sigma(B) \subset \Omega$, and any $\lambda \in [0, 1]$ satisfy the inequality $f(\lambda A + (1-\lambda)B) \le \lambda f(A) + (1-\lambda) f(B)$.

We denote the class of all $\mathcal{A}$-monotone functions by $\mathfrak{P}_{\mathcal{A}}$, we do the class of all $\mathcal{A}$-convex functions by $\mathfrak{Q}_{\mathcal{A}}$. Identifying, as usual, the algebra $\mathcal{B}(\mathcal{H}_n)$ of all operators in an n-dimensional Hilbert space $\mathcal{H}_n$ with a distinguished orthonormal basis with the algebra of matrices $M_n(\mathbb{C})$, we obtain for $\mathfrak{P}_{\mathcal{B}(\mathcal{H}_n)}$ and $\mathfrak{Q}_{\mathcal{B}(\mathcal{H}_n)}$ thoroughly studied classes of matrix monotone and matrix convex functions of order n; in what follows we denote them by $\mathfrak{P}_n$ and $\mathfrak{Q}_n$. Note that $\mathfrak{P}_1 \supset \mathfrak{P}_2 \supset \mathfrak{P}_3 \supset \cdots$ and, analogously, $\mathfrak{Q}_1 \supset \mathfrak{Q}_2 \supset \mathfrak{Q}_3 \supset \cdots$, where all inclusions are strict (see, e.g., [2] for relevant examples). For all infinite-dimensional Hilbert spaces $\mathcal{H}$, independently of cardinalities of their bases, the corresponding classes $\mathfrak{P}_{\mathcal{B}(\mathcal{H})}$ and $\mathfrak{Q}_{\mathcal{B}(\mathcal{H})}$ appear to be the same. We denote them by $\mathfrak{P}_\infty$ and $\mathfrak{Q}_\infty$. It is well-known [3] that

$$\mathfrak{P}_\infty = \bigcap_{n=1}^{\infty} \mathfrak{P}_n, \quad \mathfrak{Q}_\infty = \bigcap_{n=1}^{\infty} \mathfrak{Q}_n. \tag{1}$$

*E-mail: dinhtrunghoa@yahoo.com.
**E-mail: Oleg.Tikhonov@ksu.ru.