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ON ALGEBRAIC PROPERTIES OF SETS WHICH DO NOT CONTAIN STRAIGHT LINES

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1. Main results

Let A be a set in an n-dimensional Euclidean space E_n, where $n > 1$, and $\Gamma(A)$ its boundary. Assume that A is contained in a convex set which does not contain straight lines (e. g., A is bounded). In this article we prove (see Theorem 3 below) that for such a set the following inclusion then will be fulfilled

$$2A \subseteq \Gamma(A) + \Gamma(A), \tag{1.1}$$

where "+" stands for the algebraic (vector) sum and $2A$ is the set of vectors from A multiplied by 2. Inclusion (1.1) can be rewritten in the form

$$A \subseteq \frac{\Gamma(A) + \Gamma(A)}{2},$$

which means that each point of the set A can be represented as a half-sum of some two boundary points of A.

We also prove (see Theorem 5 below) that, under the additional assumption that A is connected, inclusion (1.1) can be sharpened as follows:

$$A + A \subseteq \Gamma(A) + \Gamma(A). \tag{1.2}$$

It is of interest to note that, for $n = 1$, generally speaking, inclusions (1.1) and (1.2) do not hold. To this end it suffices to take in E_1 either a ray or a segment (which is not degenerating into a point). Inclusions (1.1) and (1.2) may also fail to hold for convex sets containing straight lines. To verify this, one can consider a half-plane in E_2.

In Section 3 we prove inclusions (1.1) and (1.2) by applying a technique of "discretization of the space". Namely, the space E_n is divided by hyperplanes, parallel to the coordinate ones, into cubes with the length of edge equaling δ, and the initial set A is approximated by a set which is simpler by its nature. Namely, we use a set consisting of above cubes (these are the sets $\Omega^{0\delta}$ and Ω^{δ} in Theorems 3 and 5, respectively). In fact, inclusions (1.1) and (1.2) are first proved for approximating sets already constructed and afterwards, by means of passing to limit as $\delta \to 0$, the necessary inclusions are expanded also onto the initial set A.

In Section 2, to realize the mentioned approach, we introduce a series of notions and state a series of auxiliary propositions (some of them, e. g., Theorems 1 and 2, are of an autonomous interest).

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