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Becker Type Univalence Conditions for Harmonic Mappings

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Abstract—We obtain Becker type univalence conditions for locally univalent harmonic mappings defined in one of the following domains: the unit disc, a halfplane, the exterior of the unit disc and prove a generalization of John's univalence condition.

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Introduction. The aim of this paper is to construct new families of univalent harmonic mappings of three domains: the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$, a half-plane, and the domain $D^- = \{\zeta \in \overline{\mathbb{C}} : |\zeta| > 1\}$. Namely, we use the method by L. Ahlfors and G. Weill [1] to generalize the following assertions due to J. Becker (see [2–4]).

Theorem B$_1$. *Let $f : D \to \mathbb{C}$ be a holomorphic function satisfying the condition $f'(z) \neq 0$. If $(1 - |z|^2)\,|zf''(z)/f'(z)| \leq 1$ at any $z \in D$, then the function f is univalent on D.*

Theorem B$_2$. *Let $F : D^- \to \overline{\mathbb{C}}$ be a holomorphic function on the domain $D^- \setminus \{\infty\}$ with a simple pole at the point at infinity, and let $F'(\zeta) \neq 0$ on D^-. If $\left(|\zeta|^3 - |\zeta|\right)|F''(\zeta)/F'(\zeta)| \leq 1$ at any point $\zeta \in D^-$, then the function F is univalent on D^-.*

One can find some generalizations and refinements for analytic functions of these theorems in papers [5–9]. By the way, there are the following inequalities for univalent conformal mappings $f : D \to \mathbb{C}$ and $F : D^- \to \overline{\mathbb{C}}$, related to Becker's conditions:

$$(1 - |z|^2)\left|\frac{f''(z)}{f'(z)}\right| \leq 6 \;\; \forall z \in D, \quad \left(|\zeta|^3 - |\zeta|\right)\left|\frac{F''(\zeta)}{F'(\zeta)}\right| \leq 6 \;\; \forall \zeta \in D^-.$$

These well-known inequalities are consequences of the classical results by L. Bieberbach for $f \in S$ and by G. M. Goluzin for $F \in \Sigma$.

Now, we consider harmonic mappings, defined by the formula

$$f(z) = h(z) + \overline{g(z)}, \;\; z \in D, \tag{1}$$

where the functions h and g are holomorphic in the unit disc D. According to H. Lewy's theorem (see [10, 11]), the mapping f is sense preserving and locally univalent if and only if $|h'(z)| > |g'(z)|$ at any point $z \in D$.

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