Russian Mathematics (Iz. VUZ) *Izvestiya VUZ. Matematika*
Vol. 41, No. 5, pp. 74–77, 1997 *UDC 546.17*

ON THE TOPOLOGIES OVER SEMIGROUPS AND GROUPS, WHICH ARE DEFINED BY FAMILIES OF DEVIATIONS AND NORMS

Khamza Buzhuf and V. V. Mukhin

The way to create topology on semigroups by means of deviations[1] (see [1]) is not universal in the following sense: There exist topological semigroups with a topology which cannot be made uniform. In the capacity of an example one can take a non-uniformizing topological space X with a multiplication $(x, y) \mapsto y$, which is evidently continuous. Nevertheless the question of description of families of deviations, which generate a topology on X, consistent with the semigroup operation, is of interest. A part of the results of the present article was announced by the authors in [2]. In [3], those families of deviations were defined via an invariant measure.

Let us recall that a mapping $f : X \times X \to [0, +\infty)$ is called a *deviation* on a set X if $f(x, y) = f(y, x)$ and $f(x, y) \leq f(x, z) + f(z, y)$ for arbitrary x, y, and z from X. The deviation f on a semigroup X is said to be *invariant from the left* (*from the right*) if $f(xy, xz) = f(y, z)$ $(f(yx, zx) = f(y, z))$ for arbitrary x, y, and z from X.

Every family $\{f\}$ of deviations on a set X generates in a standard way a topology on X: the sets $\{x \in X \mid f(x, y) < a\}$, where $y \in X$, $f \in \{f\}$, and $a > 0$, form a pre-base of such a topology. One can rather simply demonstrate the following

Theorem 1. *Assume that a topology τ on a semigroup X is generated by a family of deviations invariant from the left. Then the left translations $x \to ax$ $(x \in X)$ are continuous in (X, τ) for each $a \in X$.*

Clearly, an analogous theorem takes place as well for a family of deviations invariant from the right. Further we shall not dwell into similar remarks.

Theorem 2. *Let a topology τ on X be generated by the family $\{f\}$ of deviations invariant from the left. Then the following conditions are equivalent to each other*:

A) *for arbitrary $a, x \in X$, every deviation $f \in \{f\}$, and every number $\alpha > 0$, there exist deviations $f, \dots, f_n$ from $\{f\}$ and a number $\beta > 0$ such that $f(sa, xa) < \alpha$ for all $s \in X$ which obey $f_i(s, x) < \beta$ simultaneously for all $i = 1, \dots, n$*;

B) *the semigroup operation $(x, y) \mapsto xy$ is continuous*;

C) *the right translations $x \mapsto xa$ are continuous for every $a \in X$.*

If, in addition, X is a group, then each of the conditions A), B), C) *implies the continuity of the inversion.*

Proof. Let there be fulfilled the condition A). Let $x, y \in X$, $\varepsilon > 0$, $g_1, \dots, g_n \in \{f\}$ and $W = \{t \in X \mid g_i(t, xy) < \varepsilon,\ i = 1, \dots, n\}$. Then there exist $f_1, \dots, f_m \in \{f\}$ and $\beta > 0$ such that $g_1(sy, xy) < \varepsilon/2$ for every $i = 1, \dots, n$ with all $s \in X$ such that there holds $f_j(s, x) < \beta$

[1] Translator remark: The terms “pseudometrics”(by Kuratowski) and “semimetrics” (e. g., by Edwards) can be used. We keep the most correct Russian word here.