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A THEOREM ON SEPARABILITY OF A CONVEX POLYHEDRON FROM ZERO POINT OF THE SPACE AND ITS APPLICATIONS IN OPTIMIZATION

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In the theory of extremal problems the so-called separability theorems play an important role. Usually the proof of the theorems on the separability of a convex set $D \subset R_n$ and a point $x_0 \notin D$ (e. g., [1], p.200; [2]–[4]) is reduced to the construction of a hyperplane, separating them, in the form

$$\langle d, x - x_0\rangle \geq \|d\|^2 \quad \forall x \in D;$$

here $d = \mathbf{P}_D(x_0) - x_0$, $\mathbf{P}_D(x_0)$ is a projection of the point x_0 onto the set D. In this paper, we propose a new approach to the construction of the set of normal vectors to hyperplanes such that each one strongly separates a convex combination of a finite set of points from zero point of the space. We also adduce in this paper one application of the separability theorem which concerns the theory of nondifferential optimization, in particular, the problem of projecting of the zero point of the space onto a convex polyhedron by finite methods.

1^0. The separability theorem

Let $J = \{1, 2, \dots, m\}$, $z_i \in R_n$, $i \in J$ and

$$L = \Big\{z \in R_n : z = \sum_{i\in J} \alpha_i z_i,\ \alpha_i \geq 0,\ \sum_{i\in J} \alpha_i = 1\Big\}.$$

Evidently, L is a nonempty closed bounded convex set. Put

$$\Omega_i = \{y \in R_n : \langle z_i, y\rangle \geq \|y\|^2\}, \quad i \in J.$$

Lemma 1. *The set $\Omega_i \subset R_n$, $i \in J$, is an n-dimensional ball of the radius $R_i = \|z_i\|/2$ with the center at the point $O_i = z_i/2$.*

Proof. Adding the value $\|z_i\|^2/4$ to both sides of the inequality which defines the set Ω_i we obtain $\|y - z_i/2\|^2 \leq \|z_i\|^2/4$. □

Since the set Ω_i, $i \in J$, is convex, closed, and bounded, the set

$$\Omega = \bigcap_{i\in J} \Omega_i = \{y \in R_n : \langle z_i, y\rangle \geq \|y\|^2 \ \forall i \in J\}$$

is a convex compact.

Definition 1 ([1], p. 198). Let D be a set from R_n. One says that a hyperplane $\langle c, u\rangle = a$ with the normal vector $c \neq \mathbf{0}$ separates the set D from the point x_0, if $\langle c, d\rangle \geq a$ for all $d \in D$ and $\langle c, x_0\rangle \leq a$, i. e., if the following inequalities are true: $\langle c, x_0\rangle \leq a \leq \inf_{d\in D}\langle c, d\rangle$. If $\langle c, x_0\rangle < \inf_{d\in D}\langle c, d\rangle$, then one says that the set D and the point x_0 are strongly separable.