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EXACT ESTIMATES FOR THE ERROR GRADIENT OF LOCALLY ONE-DIMENSIONAL METHODS FOR MULTIDIMENSIONAL EQUATION OF HEAT CONDUCTION

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Locally one-dimensional difference methods compose an important family among the methods for solving evolution problems. A large number of works is dedicated to their construction and investigation (see [1]–[4]). However, the question of deduction of exact bounds for errors and another question, closely related to the first, on the optimality of these methods turn to be poorly investigated even for parabolic problems. At the same time, for purely implicit methods and those with splitting operator, these questions have already been well developed (see [5]–[13]).

In the present article we study two locally one-dimensional difference methods — the "consecutive" one (already classical) and that with parallelizing at auxiliary steps. Optimal estimates of errors in $L_2(Q)$ for the first method were obtained as early as in [7], while for the second one — recently in [14]. Here we shall obtain both lower and upper estimates for error in the energy norm and for error gradient in $L_2(Q)$ on the class of right sides being from $L_2(Q)$ (and initial functions being from $\overset{\circ}{W}{}_2^1(\Omega)$). The obtained estimates will coincide with those known for other methods (see [9]) only under a strict condition $\tau = O(h^2)$. But, generally speaking, they essentially lose against the known ones; in particular, if $\tau \geq \varepsilon h$, then there is *simply absent the convergence* to zero of the gradient of error in $L_2(Q)$ on the mentioned class of data. Moreover, we shall deduce the estimates of the order $O(\sqrt{\tau} + h)$ under nonstandard conditions upon f. Our results are proved by means of an original technique which develops that suggested in [7], [12], [14].

The main results are formulated in Section 1. In Section 2 one can find auxiliary assertions. The proof for the upper estimates of errors is given in Section 3, and for the lower ones — in Section 4.

1. Formulation of the main results

Let us consider the first initial-boundary value problem for the equation of heat conduction:

$$\frac{\partial u}{\partial t} - a^2 \triangle u = f(x,t) \quad \text{in } Q = \Omega \times (0,T), \tag{1.1}$$

$$u|_{\partial\Omega\times(0,T)} = 0, \quad u|_{t=0} = u_0(x) \quad \text{on } \Omega = (0,X_1)\times\cdots\times(0,X_n), \tag{1.2}$$

where $x = (x_1,\dots,x_n)$, $n \geq 2$, $\triangle = \sum_{i=1}^n \partial^2/\partial x_i^2$ is the Laplace operator, $\partial\Omega$ is the boundary of Ω. Assume below that $f = \sum_{i=1}^n f_{(i)}$ and $a = 1$ (which does not restrict the generality). Let $f \in L_2(Q)$ and $u_0 \in \overset{\circ}{W}{}_2^1(\Omega)$.

We introduce the steps $h_1 = X_1/N_1, \dots, h_n = X_n/N_n$ and $\tau = T/M$, where $N_m \geq 2$, $m = \overline{1,n}$, and $M \geq 2$. Let $h = (h_1,\dots,h_n)$, $|h| = (h_1^2 + \cdots + h_n^2)^{1/2}$. We introduce in $\overline{\Omega}$ a uniform grid

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