Russian Mathematics (Iz. VUZ) *Izvestiya VUZ. Matematika*
Vol. 41, No. 5, pp.30–37, 1997 *UDC 517.929*

THE VALÉE-POUSSIN THEOREM FOR A CLASS OF FUNCTIONAL-DIFFERENTIAL EQUATIONS

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Consider a two-point boundary value problem

$$(\mathcal{L}x)(t) \equiv x^{(n)}(t) + \int_a^b x(s)d_sR(t,s) = f(t), \quad t \in [a,b], \tag{1}$$

$$x^{(i)}(a) = 0, \quad i = \overline{0, n-k-1}; \quad x^{(j)}(b) = 0, \quad j = \overline{0, k-1}; \quad k \in \{1, 2, \dots, n-1\}, \tag{2}$$

where $R : [a,b] \times [a,b] \to R^1$ is a measurable function, the complete variation $\bigvee_{s=a}^{b} R(t,s)$ and $f(t)$ being summable on $[a,b]$.

It is well-known [1] that particular cases of equation (1) are equations with deviating argument, integro-differential equations, and their "hybrids".

We denote by W^n the Banach space of functions $x : [a,b] \to R^1$ having an absolutely continuous derivative $x^{(n-1)}(t)$,

$$\|x\|_{W^n} = \int_a^b |x^{(n)}(s)|ds + \sum_{i=1}^{n-1} |x^{(i)}(a)|.$$

By a *solution of equation* (1) we shall call a function $x \in W^n$ which satisfies the equation almost everywhere on $[a,b]$.

Different aspects of both equation (1) and problem (1), (2) were studied in [2], [3], and [4]. Consider the question about the conditions for the sign-preserving of the Green function of problem (1), (2).

It is well-known that (see [1]) if problem (1), (2) has a unique solution for any summable function $f(t)$, then the solution can be written as follows:

$$x(t) = \int_a^b G(t,s)f(s)ds, \tag{3}$$

where the kernel $G(t,s)$ is called the *Green function* of the problem. Representation (3) implies that $G(t,s)$ is not uniquely defined, since, for a fixed $t \in [a,b]$, the variation of values of $G(t,\cdot)$ over a zero measure set does not change the value of $\int_a^b G(t,s)f(s)ds$. Therefore, the assertion about the sign of $G(t,s)$ should be understood as follows: The class of equivalent kernels $G(t,s)$ contains a function satisfying this condition.

1. Let k be an odd integer. We can represent a function $R(t,s)$, which has a bounded variation for almost all $t \in [a,b]$, as the following difference:

$$R(t,s) = R^+(t,s) - R^-(t,s), \tag{4}$$

where $R^+(t,s)$ and $R^-(t,s)$ do not decrease with respect to s for almost all $t \in [a,b]$.