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ON CLONES OF POLYNOMIALS OVER INFINITE FIELDS

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1. Introduction

In this article we continue the studies started in [1]. It is devoted to the problem of classification of a part of closed classes of polynomials over infinite fields.

Let us introduce basic notions. Let a class of functions which map Cartesian powers of a set K to K be given. This class is said to be closed if it is closed with respect to superpositions. More precisely, given an m-ary function f and n-ary function g, we define the r-ary function $h = f * g$ $(r = m + n - 1)$ by

$$h(x_1, \dots, x_r) = f(g(x_1, \dots, x_n), x_{n+1}, \dots, x_r),$$

where $x_1, \dots, x_r \in K$. Next, we define the operations ζ, τ, Δ and ∇ by setting for $m > 1$

$$\begin{aligned}(\zeta f)(x_1, \dots, x_m) &= f(x_2, \dots, x_m, x_1),\\ (\tau f)(x_1, x_2, x_3, \dots, x_m) &= f(x_2, x_1, x_3, \dots, x_m),\\ (\Delta f)(x_1, \dots, x_{m-1}) &= f(x_1, x_1, x_2, \dots, x_{m-1}),\\ (\nabla f)(x_1, \dots, x_{m+1}) &= f(x_2, \dots, x_{m+1})\end{aligned}$$

for all $x_1, \dots, x_{m+1} \in K$ and $\zeta f = \tau f = \Delta f = f$ for $m = 1$. A class of functions C is said to be closed if it is closed with respect to the operations $*$, ζ, τ, Δ, and ∇. This approach due to A.I. Mal'tsev allows to consider closed classes as algebras with respect to the introduced operations (see [2]). A closed class is called a clone if it contains all the projections, i. e., the functions of the form $e_i^n(x_1, \dots, x_n) = x_i$, $1 \le i \le n$.

Now let K be an arbitrary associative commutative ring with identity. We consider algebraic polynomials over K in arbitrary (finite) number of variables and closed classes consisting of such polynomials. We shall use the following notation:

F_K is the class of all polynomials over K;

F_K^0 is the class of all polynomials over K without constant term;

L_K is the class of all linear polynomials over K;

L_K^0 is the class of all linear polynomials over K without constant term.

We pose the problem of classification of clones belonging to the interval $[L_K^0, F_K]$ of the lattice of all clones. We understand this problem as clarifying the lattice structure of the indicated interval, which is depicted in Diagram 1, where the question marks signify that the corresponding intervals may contain closed classes which do not coincide with the four ones indicated above. This depends on the properties of the ring K.

In [3], it was shown that, in the case of a residue class ring modulo any number, the lattice interval $[L_K^0, F_K]$ is the subdirect product of the intervals $[L_K^0, F_K^0]$ and $[L_K^0, L_K]$. The proof remains true also for an arbitrary ring K. Thus, the investigation of the interval $[L_K^0, F_K]$ can be reduced, in a

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