ISSN 1066-369X, Russian Mathematics , 2017, Vol. 61, No. 1, pp. 53–68. © Allerton Press, Inc., 2017.
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Rings of Quasi-Endomorphisms of Some Direct Sums of Torsion-Free Abelian Groups

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Received June 17, 2015

Abstract—We consider a representation of quasi-endomorphisms of Abelian torsion-free groups of rank 4 by matrices of order 4 over the field of rational numbers $\mathbb{Q}$. We obtain a classification for quasi-endomorphism rings of Abelian torsion-free groups of rank 4 quasi-decomposable into a direct sum of groups A_1, A_2 of rank 1 and strongly indecomposable group B of rank 2 such that quasi-homomorphism groups $\mathbb{Q} \otimes \mathrm{Hom}(A_i, B)$ and $\mathbb{Q} \otimes \mathrm{Hom}(B, A_i)$ for any $i = 1, 2$ have rank 1 or are zero. Moreover, for algebras from the classification we present necessary and sufficient conditions for their realization as quasi-endomorphism rings of these groups.

DOI: 10.3103/S1066369X17010078

Keywords: *ring of quasi-endomorphisms, Abelian group, torsion-free group, quasi-decomposable group.*

1. INTRODUCTION

The concept of the Abelian Groups quasi-homomorphisms was introduced by B. Johnson [1] in connection with the discovery of the anomaly in direct expansions of finite rank Abelian groups. B. Johnson [2] constructed an example of a rank 4 Abelian torsion-free group, that both is decomposable into a direct sum of two rank 2 irreducible groups, and has a decomposition into a direct sum of indecomposable groups of ranks 1 and 3. Replacement of the homomorphism concept with the weaker quasi-homomorphism concept led to the emergence of a category of finite rank Abelian group quasi-homomorphisms, where objects are torsion-free finite rank groups, and morphisms are the quasi-homomorphisms. In this category, any object uniquely decomposes into a direct sum of indecomposable objects. This decomposition into a direct sum is called a quasilinear expansion and the irreducible objects are said to be strongly indecomposable groups.

At the same time there is a problem of $\mathbb{Q}$-algebras classification, i.e., the algebras over the field of rational numbers $\mathbb{Q}$, realized as quasi-endomorphism algebras of Abelian small rank torsion-free groups. J. Reid found [3] a connection between the structure of a finite rank torsion-free group and its quasi-endomorphism algebra thus reinforcing the problem urgency. Its solution for groups of rank 1 follows from the results by P. Baer [4]. It turns out that the rational number ring is the only algebra that can be realized as the algebra of rank 1 group quasi-endomorphisms.

Note that here and in what follows, the word "group" stands for an additive Abelian torsion-free group of finite rank.

P. Beaumont and P. Pierce [5] proved that the following five algebras and one infinite algebra series, and only they are up to isomorphism the algebras of rank 2 groups quasi-endomorphisms: $\mathbb{Q}$, $\mathbb{Q} \oplus \mathbb{Q}$, quadratic field, the ring of 2×2 matrices over $\mathbb{Q}$, the ring of the lower triangular 2×2 matrices over $\mathbb{Q}$ with equal diagonal elements, the ring of all lower triangular 2×2 matrices over $\mathbb{Q}$.

The classification problem for rank 3 groups was solved by the author in [6–8]. We proved that there exist up to isomorphism or anti-isomorphism eighteen algebras and four infinite algebra series that are

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