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INEQUALITIES OF BERNSHTEĬN AND JACKSON–NIKOL'SKIĬ TYPE AND THEIR APPLICATIONS

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1. Necessary definitions and statement of problem

Let $\pi_s = [-\pi, \pi]^s$ be an s-dimensional cube, $L^p(\pi_s)$, $1 \le p < \infty$, a set of all measurable functions $f(x) = f(x_1, \dots, x_s)$, 2π-periodic with respect to each variable, and such that

$$\|f\|_p = (2\pi)^{-s}\Big(\int_{\pi_s} |f(x)|^p dx\Big)^{\frac{1}{p}} < \infty,$$

$$L_0^p(\pi_s) = \Big\{f \in L^p(\pi_s) : \int_{-\pi}^{\pi} f(x)dx_j = 0,\ j = 1, \dots, s\Big\}.$$

For a subset B of the Euclidean space R^s we denote by B_0 and B_+ sets consisting of all elements $x = (x_1, \dots, x_s) \in B$, whose every component is nonnegative or positive, respectively. As usual, we denote by Z^s the integer-valued lattice of R^s. For $n \in Z_+^s$, we put $\|n\|_1 = n_1 + \dots + n_s$, $2^{-n} = (2^{-n_1}, \dots, 2^{-n_s})$.

For $f \in L^p(\pi_s)$, we introduce the mixed module of smoothness of order $k \in Z_+^s$:

$$\Omega_k(f;t)_p \equiv \Omega_k(f;t_1, \dots, t_s)_p = \sup_{\substack{|h_j| \le t_j \\ j=1,\dots,s}} \|\Delta_h^k f(x)\|_p, \quad t \in [0,1]^s,$$

where $\Delta_h^k f(x) = \Delta_{h_s}^{k_s} \dots \Delta_{h_1}^{k_1} f(x)$, $\Delta_{h_j}^{k_j} = \Delta_{h_j}^1(\Delta_{h_j}^{k_j-1})$, $\Delta_{h_j}^1 f(x) = f(x_1, \dots, x_j + h_j, \dots, x_s) - f(x_1, \dots, x_j, \dots, x_s)$.

For $f \in L^p(\pi_s)$, we denote by $E_G(f)_p$ the best approximation of the function f by polynomials from $T(G)$, where G is a finite set of points from Z^s, and

$$T(G) = \Big\{t(x) : t(x) = \sum_{n \in G} c_n e^{i(n,x)}\Big\}.$$

In this article the spectrum G is given by a continuous on $[0,1]^s$ function $\Lambda(t) = \Lambda(t_1, \dots, t_s)$, which decreases with respect to every variable if other variables are fixed and is such that $\Lambda(t) > 0$ ($\Lambda(t) = 0$) for $\prod_{j=1}^{s} t_j > 0$ $\Big(\prod_{j=1}^{s} t_j = 0\Big)$.

For $N > 0$, we define the sets

$$\Gamma(\Lambda, N) = \{m \in Z_+^s : \Lambda(2^{-m}) \ge \tfrac{1}{N}\}, \quad \Gamma^{\perp}(\Lambda, N) = Z_+^s \setminus \Gamma(\Lambda, N),$$

$$\rho(n) = \{m = (m_1, \dots, m_s) \in Z^s : 2^{n_j - 1} \le |m_j| < 2^{n_j}\} \ \ (n \in Z_+^s),$$

$$Q(\Lambda, N) = \bigcup_{n \in \Gamma(\Lambda, N)} \rho(n).$$