

ON SOME INDICATORS OF THE QUASI-NILPOTENCY OF FUNCTIONAL OPERATORS

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Usually, an immediate application of the known I.M. Gel'fand's spectral radius formula to the verification whether linear bounded operators (l. b. o.) are quasi-nilpotent, turns out to be significantly difficult in view of an awkwardness of the operator or its majorant. Therefore some special tests indicating the quasi-nilpotency represent a certain interest. Such tests (see, e. g., [1]–[4]) are frequently related to a certain version of the concept of the Volterra functional operators; for instance, for the definition and properties of the abstract Volterra operators see [5], [6]. In what follows we shall give applicable sufficient conditions for the quasi-nilpotency of functional operators. For these conditions we shall use the Volterra property concept in the sense of [7] and [8], which is a direct generalization of the known definitions in [9], [10] for the case of several variables (see also [6]). Here we shall develop some results in [1]. Some propositions given in the present article amplify the results in [11], [12].

Let us restrict ourselves to the case, important for applications, of the operators acting in the space L_∞ . Let $\Pi \subset \mathbf{R}^n$ be a Lebesgue-measurable and bounded set, $L_\infty = L_\infty(\Pi)$. $\Sigma = \Sigma(\Pi)$ means the σ -algebra of Lebesgue-measurable subsets in Π , and T is a certain part of Σ . In accordance with [7] and [8], we call an l. b. o. $G[\cdot] : L_\infty \rightarrow L_\infty$ by the *Volterra operator on the system of sets* T if $\forall H \in T$ the restriction $G[x]|_H$ does not depend on the values of the restriction $x|_{\Pi \setminus H}$, i. e., $\forall H \in T$ we have $P_H G P_H = P_H G$, where P_H is the operator of multiplication by the characteristic function of the set H . We denote by $V(T)$ the class of all Volterra operators on the system T . Let $[a, b] = [a^1, b^1] \times \cdots \times [a^n, b^n]$ be a fixed bar containing Π , and $\mu = \{\nu, \lambda\}$ any ordered pair of number collections $\nu, \lambda \subset \{1, \dots, n\}$, $\nu \cup \lambda \neq \emptyset$, $\nu \cap \lambda = \emptyset$. We put $T_\mu(\Pi) \equiv \{\Pi \cap [\alpha, \beta] \mid [\alpha, \beta] \in T_\mu([a, b])\}$, where

$$T_\mu([a, b]) \equiv \{[\alpha, \beta] = [\alpha^1, \beta^1] \times \cdots \times [\alpha^n, \beta^n] \subset [a, b] \mid \alpha^i = a^i \ (i \notin \nu), \ \beta^i = b^i \ (i \notin \lambda)\}.$$

A system of sets $\mathcal{T} = \{H_0, \dots, H_k\} \subset \Sigma$ is termed a *finite chain* if it is ordered by the inclusion ($H_0 \subset H_1 \subset \cdots \subset H_k$) and satisfies the conditions $H_0 = \emptyset$, $H_k = \Pi$. If, in addition, $G \in V(\mathcal{T})$, then we shall say that \mathcal{T} is a finite Volterra chain of the operator G . Let $\delta > 0$ be a fixed value, $CH = \Pi \setminus H$. A finite chain $\mathcal{T} = \{H_0, \dots, H_k\}$ is termed a *weak δ -chain of l. b. o.* $G : L_\infty \rightarrow L_\infty$ if $\rho(P_{H_i \setminus H_{i-1}} G P_{H_i \setminus H_{i-1}}) < \delta$, $i = \overline{1, k}$. We denote by $\Psi_{\text{weak}}(G)$ a set of all nonnegative numbers δ such that for each δ the Volterra weak δ -chain of the operator G exists. For the space L_∞ under consideration the following generalization of theorem 1 in [1] is valid (this generalization was proved in [12]): For any l. b. o. $G : L_\infty \rightarrow L_\infty$, the spectral radius $\rho(G)$ equals $\inf \Psi_{\text{weak}}(G)$.

We say that an operator $G : L_\infty \rightarrow L_\infty$ satisfies the δ -condition on the set $H \in \Sigma$ if $\|P_H G P_H\| < \delta$. A finite chain $\mathcal{T} = \{H_0, \dots, H_k\}$ will be called the δ -chain of the operator $G : L_\infty \rightarrow L_\infty$ if G satisfies the δ -condition on each difference $H_i \setminus H_{i-1}$, $i = \overline{1, k}$. Let $\Psi(G)$ be a set of all nonnegative numbers δ such that the Volterra δ -chain of the operator G exists. Obviously, $\Psi(G) \subset \Psi_{\text{weak}}(G)$.

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