A Stinespring Type Representation for Operators in Hilbert Modules Over Local $C^\star$-Algebras

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Abstract—We prove an analog of the Stinespring theorem for Hilbert modules over local $C^\star$-algebras.

INTRODUCTION

Hilbert modules over involutive topological algebras represent far-reaching generalizations of the Hilbert space. For the first time Hilbert modules over commutative $C^\star$-algebras were considered by I. Kaplansky in [1]. Now the theory of Hilbert modules over $C^\star$-algebras is an actively developing domain of mathematics, which is reflected in monographs [2, 3].

In the second half of the XX-th century in papers [4−6] one studied more general topological involutive algebras and modules over them. There is some progress in studying structural properties of this class of modules [7]. One of important classes of linear maps acting between topological involutive algebras A and B is presented by *completely positive* operators $T : A \to B$ which map positive elements of $M_n(A)$ to positive elements of $M_n(B)$ for any $n \in \mathbb{N}$, where $M_n(A)$ means the algebra of square matrices $n \times n$ with elements from A. Completely positive maps are widely applied in the quantum theory of information [8]. One of the most deep results of the theory of completely positive maps is the theorem proved by P. Stinespring ([9], theorem 1). According to this theorem, the study of a completely positive map from the $C^\star$-algebra A to the $\mathcal{L}(\mathcal{H})$-algebra of linear bounded operators of a Hilbert space $\mathcal{H}$ can be reduced (in a sense) to studying a ($\star$)-homomorphism acting from the algebra A to $\mathcal{L}(\mathcal{H}')$, where $\mathcal{H}'$ is also a Hilbert space. Later the Stinespring result was generalized to the case of Hilbert modules over $C^\star$-algebras [10] and more general topological algebras [11]. An interesting variant of the Stinespring theorem has been presented recently in papers [12, 13]. The goal of this paper is to extend the results obtained in [12, 13] to the case of Hilbert modules over local $C^\star$-algebras.

1. PRELIMINARY INFORMATION

Let us give some information necessary for further reasoning. The goal of this Section is to define the terminology and denotations and to introduce the necessary notions. See [14−16] for complete information about $C^\star$-algebras and more general topological algebras with involution. All algebras are considered over a field of complex numbers.

An algebra with an involution A is said to be an involutive *LMC-algebra*, if A is a locally convex topological vector space, where the topology is given by the set of seminorms $(p_\lambda)_{\lambda \in \Lambda}$ satisfying the following conditions:

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