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Perturbation of the Hill Operator by Narrow Potentials

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Abstract—We consider a perturbation of a periodic second order differential operator, defined on the real axis, which is a special case of the Hill operator. The perturbation is realized by a sum of two complex-valued potentials with compact supports. The potentials depend on two small parameters. One of them describes the lengths of the supports of the potentials and the reciprocal to the second one corresponds to the maximum values of the potentials. We obtain a sufficient condition, under fulfillment of which, the eigenvalues arise from the edges of non-degenerate lacunas of continuous spectrum, and construct their asymptotics. We also give a sufficient condition under which the eigenvalues do not arise.

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INTRODUCTION

In the quantum solid state physics, the operator $-\frac{d^2}{dx^2}+q+W$, $x\in\mathbb{R}$, is used; here W is a real-valued function and q is a real-valued periodic function. It is the operator of the energy of an electron in the one-dimensional model of a crystal (see, e.g., [1]). The equation $\big(-\frac{d^2}{dx^2}+q+W\big)\psi=E\psi$ arises as a model for the Bloch electron in a crystal placed in an external electric field. The function q defines the crystal lattice and corresponds to the effective field in the crystal. The function W describes the external homogeneous electric field or the field of impurities. The Hill operator

$$\mathcal{H}:=-\frac{d^2}{dx^2}+q \tag{1}$$

matches to the pure, i.e., without impurities, crystal; the crystal is idealized and infinite.

In [2] the operator $\mathcal{H}+W$ is considered with a 1-periodic real-valued function q. The perturbation W is a real-valued non-periodic function. It is shown that, under the condition $\int\limits_{\mathbb{R}}|W(x)|(1+|x|)dx<\infty$, in every lacuna of the continuous spectrum of the operator $\mathcal{H}+W$ there is at most a finite number of eigenvalues. Moreover, every sufficiently far lacuna contains at most two eigenvalues. It is also shown that the eigenvalues cannot lie in the continuous spectrum and the endpoints of the lacunas are not eigenvalues.

In [3, 4] they consider the operator $\mathcal{H}+W$ with the following assumptions: q is a 1-periodic piecewise continuous real-valued function, and W is a piecewise continuous real-valued function satisfying $\int\limits_{\mathbb{R}}|W(x)|(1+x^2)dx<\infty$. It was shown that, under some conditions, the perturbation can add at most a finite number of eigenvalues in every lacuna. Under any arbitrary weak perturbation of a constant

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