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EXISTENCE AND UNIQUENESS OF A POSITIVE SOLUTION OF THE BOUNDARY VALUE PROBLEM FOR CERTAIN NONLINEAR FUNCTIONAL-DIFFERENTIAL EQUATION OF THE SECOND ORDER

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In this paper, using the theory of semi-ordered spaces, we establish the existence and uniqueness conditions for a positive solution of a certain nonlinear functional-differential equation of the second order.

Let us use the following notation: C is the space $C[0,1]$, L_p $(1 < p < \infty)$ is the space $L_p(0,1)$, and W^2 is the space of functions which are defined on $[0,1]$ and have absolutely continuous derivatives.

Consider the boundary value problem

$$x''(t) + g(t)(T_1x)(t) + a(t)(T_2x)^{\frac{p}{q}}(t) = 0, \quad 0 < t < 1, \quad p, q \in (1, \infty), \tag{1}$$

$$\begin{aligned} \alpha_{11}x(0) + \alpha_{12}x(1) + \beta_{11}x'(0) + \beta_{12}x'(1) = 0, \\ \alpha_{21}x(0) + \alpha_{22}x(1) + \beta_{21}x'(0) + \beta_{22}x'(1) = 0, \end{aligned} \tag{2}$$

where α_{ij}, β_{ij} $(i,j = 1,2)$ are real numbers, $g(t)$ and $a(t)$ stand, correspondingly, for a nonnegative and nonpositive summable function, $T_i : C \to L_p$ $(i = 1,2)$ are linear positive continuous operators.

We understand a positive solution of problem (1)–(2) as a function $x \in W^2$ which is positive on the interval $(0,1)$ and satisfies almost everywhere equation (1) and boundary conditions (2).

Consider the following integral equation which is equivalent to problem (1)–(2):

$$x(t) = \int_0^1 G(t,s)g(s)(T_1x)(s)ds + \int_0^1 G(t,s)a(s)(T_2x)^{\frac{p}{q}}(s)ds, \quad 0 \le t \le 1, \tag{3}$$

where $G(t,s)$ is the Green function of the operator $-\frac{d^2}{dt^2}$ with boundary data (2).

Introduce the notation

$$\alpha \equiv \frac{\alpha_{12} + \beta_{11} + \beta_{12}}{\alpha_{11} + \alpha_{12}}, \quad \beta \equiv \frac{\alpha_{22} + \beta_{21} + \beta_{22}}{\alpha_{21} + \alpha_{22}}.$$

According to [1], under the assumptions

A) $\alpha \ne \beta$, $\alpha_{11} + \alpha_{12} \ne 0$, $\alpha_{21} + \alpha_{22} \ne 0$,

B) $\dfrac{1}{\beta - \alpha}\left[\dfrac{\beta_{21}}{\alpha_{21} + \alpha_{22}}(1 - \alpha) - \dfrac{\beta_{11}}{\alpha_{11} + \alpha_{12}}(1 - \beta)\right] \le 0$,

C) $\dfrac{1}{\beta - \alpha}\left[\dfrac{\alpha_{21} - \beta_{21}}{\alpha_{21} + \alpha_{22}}(1 - \alpha) - \dfrac{\alpha_{11} - \beta_{11}}{\alpha_{11} + \alpha_{12}}(1 - \beta)\right] > 0$,

D) $\dfrac{1}{\beta - \alpha}\left[\dfrac{\beta_{21}}{\alpha_{21} + \alpha_{22}}\alpha - \dfrac{\beta_{11}}{\alpha_{11} + \alpha_{12}}\beta\right] > 0$,

E) $\dfrac{1}{\beta - \alpha}\left[\dfrac{\alpha_{21}}{\alpha_{21} + \alpha_{22}}\alpha - \dfrac{\alpha_{11}}{\alpha_{11} + \alpha_{12}}\beta + 1\right] < 0$,

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