Russian Mathematics (Iz. VUZ) *Izvestiya VUZ. Matematika*
Vol. 47, No. 7, pp.45–47, 2003 *UDC 517.983*

SOLVABILITY OF A BOUNDARY VALUE PROBLEM FOR A NONNEGATIVELY HAMILTON SYSTEM IN A HILBERT SPACE

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From the optimality conditions for the problem of minimization of a quadratic functional on trajectories of a linear system, we obtain the nonnegatively Hamilton system of the following type:

$$\begin{aligned} \dot{x}(t) &= C(t)x(t) + S(t)y(t), \\ \dot{y}(t) &= W(t)x(t) - C^*(t)y(t) \end{aligned} \tag{1}$$

with the boundary value conditions

$$x(0) = x^0, \quad y(T) = -Vx(T). \tag{2}$$

Here, $x(t), y(t) \in H$, and H is a Hilbert space with a scalar product $\langle \cdot, \cdot \rangle$, $t \in [0, T]$, $T > 0$ and $x^0 \in H$ are fixed. The asterisk in denotation of the operator indicates the adjoint operator; $C(t), S(t), W(t), andV \in L(H)$; operators $S(t)$, $W(t)$ ($t \in [0, T]$) and V are self-adjoint nonnegative operators. All operators are assumed to be continuous in t.

In this paper, we establish the unique solvability of problem (1), (2).

By changing the variable $x^1(t) = x(t) - x^0$, $x^2(t) = y(t) + Vx^0$, one can easily prove that solvability of problem (1), (2) can be investigated by analyzing solvability of a system with a nonzero right-hand side and homogeneous boundary value conditions, namely, the following one:

$$\begin{aligned} \dot{x}^1(t) &= C(t)x^1(t) + S(t)x^2(t) + g^1(t), \\ \dot{x}^2(t) &= W(t)x^1(t) - C^*(t)x^2(t) + g^2(t), \\ x^1(0) &= 0, \quad x^2(T) = -Vx^1(T), \end{aligned} \tag{3}$$

where $g^1(t) = (C(t) - S(t)V)x^0$, $g^2(t) = (W(t) + C^*(t)V)x^0$.

Let us introduce the operator

$$(Ax)(t) = \begin{pmatrix} \frac{d}{dt} - C(t) & -S(t) \\ -W(t) & \frac{d}{dt} + C^*(t) \end{pmatrix} \begin{pmatrix} x^1(t) \\ x^2(t) \end{pmatrix}, \tag{4}$$

which acts in the space $L_2([0, T]; H \dotplus H)$ and defined on the set $D(A)$ of all absolutely continuous functions $x = x(t) = \binom{x^1(t)}{x^2(t)}$ ($x^i(t) \in H$, $i = 1, 2$) which admit derivatives in $L_2([0, T]; H \dotplus H)$ and satisfy condition (3).

Lemma 1. *No sequence $\{x_n\}$ exists such that $x_n \in D(A)$, $\|x_n\|_{L_2} = 1$ for all $n \in \mathbb{N}$ and $\|Ax_n\|_{L_2} \to 0$.*

The work was supported by Russian Foundation for Basic Research, grant no. 02-01-00351.